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BESSEL IDENTITIES IN THE WALDSPURGER CORRESPONDENCE OVER THE REAL NUMBERS

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ABSTRACT

We prove certain identities between Bessel functions attached to irreducible unitary representations of $PGL_2(\mathbf{R})$ and Bessel functions attached to irreducible unitary representations of the double cover of $SL_2(\mathbf{R})$. These identities give a correspondence between such representations which turns out to be the Waldspurger correspondence. In the process we prove several regularity theorems for Bessel distributions which appear in the relative trace formula. In the heart of the proof lies a classical result of Weber and Hardy on a Fourier transform of classical Bessel functions. This paper constitutes the local (real) spectral theory of the relative trace formula for the Waldspurger correspondence for which the global part was developed by Jacquet.

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1. Introduction

A classical result of Hardy and Weber states that the Fourier transform of the function $x^{-1/2}J_\nu(x^{1/2})$ (defined for $x > 0$) is given up to some exponential factor by $x^{-1/2}J_{\nu/2}(ax^{-1})$. Here J_ν is the classical J-Bessel function and a is a constant depending on the exponential character defining the Fourier transform (see Appendix 3 for the precise statement). In this paper we will show that this identity can be used to realize the Shimura–Waldspurger correspondence between representations of $PGL_2(\mathbf{R})$ and genuine representations of the double cover of $SL_2(\mathbf{R})$. Roughly speaking, the Shimura correspondence maps cusp forms of weight $k+1/2$ to cusp forms of weight $2k$ where k is a positive integer. The Shintani–Waldspurger correspondence is the inverse map. A cusp form of weight $2k$ (and trivial central character) has an infinite component which is a discrete series representation of $PGL_2(\mathbf{R})$. The Bessel function of this discrete series is related to $x^{-1/2}J_{2k-1}(x^{1/2})$. The infinite component of a cusp form of weight $k+1/2$ is a discrete series representation of the double cover of $SL_2(\mathbf{R})$

whose Bessel function is related to $x^{-1/2}J_{k-1/2}(ax^{-1})$. Hence the Hardy–Weber identity realizes the Shimura–Waldspurger correspondence!

More precisely, the correspondence will be realized by identities between Bessel functions attached to these representations. These functions will be functions on the group whose form will involve classical Bessel functions.

This result fits into the theory of the relative trace formula developed by Jacquet and constitutes the local ("real") spectral theory that complements the global theory in [11]. Completely analogous results for the p-adic case were obtained in [4]. This is the first study of the local ("real") spectral theory of the relative trace formula from the point of view of Bessel function identities. This theory and the corresponding p-adic theory are used in [3] to study the central value of PGL_2 automorphic L-functions.

We now give a sample of the Bessel identities that we obtain. Let

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{R} \right\}, \quad A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbf{R}^* \right\}.$$

Let π be an infinite-dimensional unitary irreducible representation of $G = GL_2(\mathbf{R})$ with a trivial central character (that is, a representation of $PGL_2(\mathbf{R})$). Let $\psi_1(x) = e^{2\pi i x}$, which we view as a character on N. We attach to π a function i_{π,ψ_1} on G which is left invariant by A and right (ψ_1, N) equivariant. i_{π,ψ_1} is real analytic on an open set in G (basically, the open Bruhat cell). Let σ be an irreducible genuine unitary representation of $\bar{S}$, the double cover of $SL_2(\mathbf{R})$. We attach to σ a function j_{σ,ψ_1} on $\bar{S}$ which is left and right (ψ_1, N) equivariant. j_{σ,ψ_1} is real analytic on the open Bruhat cell in $\bar{S}$. Our main theorem of this paper (see Theorem 19.2) is the following:

THEOREM 1.1: *Let π be as above. There exists a unique $\sigma = \sigma(\pi)$ as above such that for every $x > 0$ we have*

$$(1.1) \qquad i_{\pi,\psi_1}\begin{pmatrix} x/4 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\sqrt{2}e^{\pi i/4}\epsilon(\pi, 1/2)}{L(\pi, 1/2)} e^{4\pi i/x} x^{1/2} j_{\sigma,\psi_1}\begin{pmatrix} 0 & -x \\ x^{-1} & 0 \end{pmatrix}.$$

Here $L(\pi, 1/2)$ and $\epsilon(\pi, 1/2)$ are the L factor and epsilon factor attached to π by [10]. We remark that the values of i_{π,ψ_1} above determine π, hence the correspondence is one to one.

The above identities are proved via a case by case computation of i_{π,ψ_1} and j_{σ,ψ_1} (see Corollary 8.4 and Section 18). In practice we prove a more general identity between $i_{\pi,\psi}$ and $j_{\sigma,\psi'}$ for any two nontrivial characters ψ and ψ'. The correspondence $\pi \mapsto \sigma(\pi)$ is dependent on ψ' and turns out to be exactly the Waldspurger correspondence $\pi \mapsto \Theta(\pi, \psi')$. We recall that for one fixed π there are exactly two possible $\sigma(\pi)$ depending on ψ'. The image of all possible $\sigma(\pi)$ if we vary π and ψ' gives all the genuine unitary duals of $\bar{S}$ not including the Weil representations r_ψ^+ (see [22] p. 225 and Proposition 5) which are in the image of the trivial representation under the theta correspondence.

We remark that $i_{\pi,\psi}$ and $j_{\sigma,\psi}$ are attached to π and σ via Bessel distributions which are analogs of the more famous character distributions. For a general discussion on such distributions, see Appendix 4. Our strategy for obtaining the above equalities is the following. We consider four different types of distributions:

(a) Bessel distributions for $G = GL_2(\mathbf{R})$.

(b) Bessel distributions for $\bar{G}$, the double cover of G.

(c) Bessel distributions for $\bar{S}$.

(d) Relative Bessel distributions for G.

The distributions in (a), (b), (c) are all equivariant under (N, ψ) from both sides. The distributions in (d) are invariant on the left under A and equivariant on the right under (N, ψ). We prove a regularity theorem for each distribution. We recall that a regularity theorem proves that a given distribution is given by a locally integrable function. These are the functions $i_{\pi,\psi}$ and $j_{\sigma,\psi}$ mentioned above. The relations between the distributions imply that the relative Bessel function $i_{\pi,\psi}$ on G is a certain Fourier transform of the Bessel function $j_{\pi,\psi}$ on G while the Bessel function $j_{\sigma,\psi}$ on $\bar{S}$ is a restriction of a Bessel function of a representation which is induced from σ to $\bar{G}$. The advantage of starting from representations of G and $\bar{G}$ is that these representations have a Kirillov model and it is possible to describe the action of the Weyl element in the Kirillov model via a kernel formula. This is done in [5] for G. We give a proof of this formula in Appendix 2. A proof of the formula for the case of the principal series using a Mellin transform approach was obtained by Motohashi in [15]. (See also [20], chapter VII section 7.) In Section 13 we prove a new kernel formula for $\bar{G}$.

The kernel functions thus obtained induce functions on the group. Using these kernel formulas and an inner product formula for the Kirillov model, we show that the Bessel distributions for G and $\bar{G}$ (cases (a) and (b)) are given by their respective Bessel functions.

We also use the relation between the distributions in (a) and the distributions in (d) to get a relation between their Bessel functions. Using this relation we can compute the relative Bessel function $i_{\pi,\psi}$ via a Fourier transform and the Hardy–Weber identities mentioned above. We also show that the Bessel functions attached to distributions in (c) are restrictions of Bessel functions associated to distributions in (d). We compute them explicitly and the Bessel identities follow immediately.

We finish by showing that the Bessel identity (1.1) gives an identity between the distributions themselves (see Theorem 19.4). This identity is crucial for the applications to special values of L-functions.

Our paper is organized as follows. In Section 2 we fix some notations and prove a Lemma on Whittaker functions. In Section 3 we introduce the Bessel and relative Bessel distributions on G and show that they are given by functions, the Bessel and relative Bessel functions, on an open set of G. In Section 4 we give bounds on certain orbital integrals which we shall need for the local integrability of the various Bessel functions. We also give an explicit asymptotic expansion for the orbital integrals coming from the relative Bessel distributions. In Section 5 we prove a regularity theorem for the relative Bessel distributions. In Section 6 we recall the kernel formula of [5]. We extend the kernel function to a Bessel function on G and show that this function is locally integrable. In Section 7 we show that the Bessel distributions are given by the Bessel functions which were defined in Section 6. In Section 8 we prove a Fourier transform type relation between the Bessel functions and relative Bessel functions. We also give explicit formulas for these relative Bessel functions. In Section 9 we introduce some notation for $\bar{G}$, the double cover of $GL(2)$. In Section 10 we describe the unitary dual of $\bar{S}$ following [7], [21], [22]. In Section 11 we give a kernel formula in a certain model for an irreducible genuine unitary representation of $\bar{S}$. In Section 12 and Section 13 we translate this kernel formula into a kernel formula in the Kirillov model of representations of $\bar{G}$. We also use this kernel function to define a Bessel function on $\bar{G}$ and give explicit formulas for these functions. In Section 14 we prove that the Bessel functions defined in Section 13 are locally integrable. In Section 15 we provide an inner product formula in the Kirillov model of irreducible unitary representations of $\bar{G}$ and in Section 16 we use the inner product formula for irreducible representations of $\bar{G}$ to get inner product formulas for representations of $\bar{S}$. In Section 17 we use the results from previous sections to show that the Bessel distributions on $\bar{G}$ are given by Bessel functions. In Section 18 we use the results from Section 17 to show that the Bessel distributions on $\bar{S}$ are given by Bessel functions. Finally, in Section 19 we prove the Bessel identities for functions and distributions.

We also provide Appendix 1 in which we give some bounds on classical K-Bessel functions. In Appendix 2 we prove the kernel formula of Cogdell and Piatetski-Shapiro [5]. In Appendix 3 we compute the Fourier transform of certain combinations of classical Bessel functions using the results of Hardy and Weber mentioned above. In Appendix 4 we discuss Bessel-like distributions and give a summation formula for them.

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2. Notation and preliminaries

Let $G = GL_2(\mathbf{R})$ and $S = SL_2(\mathbf{R})$; let $\bar{G} = GL_2(\mathbf{R})$ and $\bar{S} = \overline{SL_2(\mathbf{R})}$ be the double covers of G and S respectively. Let $B = B_G$ be the Borel subgroup of upper triangular matrices in G and B_S the Borel subgroup of upper triangular matrices in S. Let $A = A_G$ be the subgroup of diagonal matrices. Let A_S be the subgroup of diagonal matrices in S. Then $A_S = \{s(a) : a \in \mathbf{R}^*\}$ where

$$s(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}.$$

Let

$$N = \left\{ n(y) = \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} : y \in \mathbf{R} \right\}.$$

Let $\bar{N}$ be the group of lower unipotent elements. Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$t(a) = \begin{pmatrix} a & \\ & 1 \end{pmatrix}, \quad \bar{n}(z) = \begin{pmatrix} 1 & \\ z & 1 \end{pmatrix}, \quad z(c) = \begin{pmatrix} c & \\ & c \end{pmatrix}.$$

Let $\mathfrak{g} = \mathfrak{gl}_2(\mathbf{R})$ be the Lie algebra of $GL_2(\mathbf{R})$. We will denote Lie algebra elements using capital letters. Let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Fix $\lambda \in \mathbf{R}^*$ and let ψ be a character of N given by

$$\psi(n(y)) = \psi_\lambda(y)$$

where $\psi_\lambda(y) = e^{2\pi i \lambda y}$. We will confuse ψ with ψ_λ. Let $d_1 x$ be the standard Lebesgue measure on $\mathbf{R}$ and let $dx = d_\lambda x$ be a Haar measure on $\mathbf{R}$ which is self-dual with respect to $\psi = \psi_\lambda$. Then

$$dx = d_\lambda x = |\lambda|^{1/2} d_1 x. \tag{2.1}$$

We let $d^*x = d^*_\lambda x = dx/|x|$. For $f \in L^1(\mathbf{R})$ we define the λ-Fourier transform by

$$(2.2)\qquad \hat{f}(y) = \hat{f}^\lambda(y) = \int f(x)\psi(-xy)dx = \int f(x)e^{-2\pi i\lambda xy}d_\lambda x.$$

It is easy to see that

$$(2.3)\qquad \hat{f}^1(y) = |\lambda|^{-1/2}\hat{f}^\lambda(y/\lambda).$$

Let dg be a Haar measure on $G = GL_2(\mathbf{R})$. We normalize dg so that $dg = |a|^{-2}dxd^*cd^*ady$ on the set of elements of the form $n(x)z(c)s(a)wn(y)$.

Let π be an infinite-dimensional irreducible unitary representation of $G = GL_2(\mathbf{R})$ on a Hilbert space H. Let H_∞ be the subspace of H of smooth vectors with the usual topology. Let $\psi = \psi_\lambda$ be a nontrivial character of N as above. It is well known [10] that there exists a unique up to scalar non-zero continuous linear functional L on H_∞ satisfying

$$(2.4)\qquad L(\pi(n)v) = \psi(n)L(v), \quad n \in N,\ v \in H_\infty.$$

We say that L is a ψ Whittaker functional on H_∞. Let

$$(2.5)\qquad W_v(g) = L(\pi(g)v), v \in H_\infty.$$

The following Lemma was communicated to us by David Soudry.

Lemma 2.1: *Let $v \in H_\infty$, $f, g \in C^\infty(R^*)$ and assume that $f(a)W_v(t(a))$ and $g(a)W_v(t(a))$ are in $L^1(R^*, d^*a)$. If*

$$\int f(a)W_{\pi(n)v}(t(a))d^*a = \int g(a)W_{\pi(n)v}(t(a))d^*a$$

for all $n \in N$, then $f(a)W_v(t(a)) = g(a)W_v(t(a))$ for all $a \in \mathbf{R}^*$.

Proof:

$$\int f(a)W_{\pi(n(x))v}(t(a))d^*a = \int f(a)W_v(t(a)n(x))d^*a$$
$$= \int f(a)W_v(t(a))\psi(ax)d^*a.$$

Hence we have that $f(a)W_{\pi(n(x))v}(t(a)) \in L^1(\mathbf{R}^*, d^*a)$ for all $x \in \mathbf{R}$. Thus, the Fourier transform of $|a|^{-1}f(a)W_v(t(a))$ is the same as the Fourier transform of $|a|^{-1}g(a)W_v(t(a))$. It follows that

$$f(a)W_v(t(a)) = g(a)W_v(t(a))$$

for all $a \in \mathbf{R}^*$. ∎

Let G be a Lie group, $\mathfrak{g}$ be the Lie algebra of G. Let $\phi \in C_c^\infty(G)$, $T\colon C_c^\infty(G) \to \mathbf{C}$ be a distribution and $x, g \in G, X \in \mathfrak{g}$. We define

$$\rho_r(x)(\phi)(g) = \phi(gx), \rho_l(x)(\phi)(g) = \phi(x^{-1}g), \tag{2.6}$$

$$\rho_r(x)(T)(\phi) = T(\rho_r(x^{-1})(\phi)), \rho_l(x)(T)(\phi) = T(\rho_l(x^{-1})(\phi)), \tag{2.7}$$

$$X(\phi)(g) = \frac{d}{dt}(\phi(ge^{tX}))_{t=0}, (X(T))(\phi) = T(X(\phi)). \tag{2.8}$$

3. Bessel and relative Bessel distributions

In this section we introduce the Bessel and relative Bessel distributions on $GL_2(\mathbf{R})$. We write down a formula which relates the two distributions and show that they are given by functions on certain open and dense sets. Our goal in the next few sections is to show that these distributions are given by these functions on the full group and to compute these functions explicitly. This will force us to consider certain orbital integrals which were considered by Jacquet and which arise naturally from our construction. We will introduce them here.

3.1. Normalized Whittaker functional. Let $G = GL_2(\mathbf{R})$ and let (π, H) be an infinite-dimensional irreducible unitary representation of G with trivial central character. Let $<,>$ be a G invariant nonzero inner product on H. Let $\psi = \psi_\lambda$ be a nonzero character of N and let L be a ψ Whittaker functional on H_∞. (See (2.4).) For $v \in H_\infty$ we set $W_v(g) = L(\pi(g)v)$. By [9] we can (and will) normalize L so that

$$< v_1, v_2 >= \int_{\mathbf{R}^*} W_{v_1}(t(a))\overline{W_{v_2}(t(a))}d^*a. \tag{3.1}$$

Remark 3.1: In practice we can start with any nontrivial Whittaker functional L and normalize the inner product on H_∞ via (3.1).

3.2. Normalized torus invariant functional. Let (π, H) and L be as above. It is well known that there exists a unique up to scalar nonzero continuous linear functional P on H_∞ satisfying

$$P(\pi(t(a))v) = P(v), \quad a \in \mathbf{R}^*,\ v \in H_\infty. \tag{3.2}$$

We shall normalize P in the following way. It is well known that the integral

$$\int L(\pi(t(a))v)d^*a = \int W_v(t(a))d^*a$$

converges for every $v \in H_\infty$. (This follows from the fact that W_v is rapidly decreasing at infinity and of the order $|a|^r$ for $r > 0$ when $|a|$ is small and v is a smooth vector in an irreducible unitary representation.) We define

$$P(v) = P_\pi(v) = \frac{1}{L(\pi,1/2)}\int L(\pi(t(a))v)d^*a = \frac{1}{L(\pi,1/2)}\int W_v(t(a))d^*a.$$

Then P is a nonzero continuous functional on H_∞ satisfying (3.2).

3.3. Bessel and relative Bessel distributions. It follows from Corollary 23.4 that for every continuous functional λ on H_∞ and every $f \in C_c^\infty(G)$ there exists a unique vector $v_{f,\lambda} \in H_\infty$ such that

$$\lambda(\pi(f)u) = < u, v_{f,\lambda} > \tag{3.3}$$

for all $u \in H$. Let (π, H), L and P be as above. We define the normalized Bessel distribution by

$$J_{\pi,\psi}(f) = \overline{L(v_{f,L})}. \tag{3.4}$$

We define the normalized relative Bessel distribution by

$$I_{\pi,\psi}(f) = \overline{L(v_{P,f})}. \tag{3.5}$$

Notice that $J_{\pi,\psi} = J_{L,L}$ and $I_{\pi,\psi} = J_{P,L}$ where $J_{\lambda,\beta}$ is defined in (23.5). Let $\tilde{f}(g) = f(g^{-1})$. It follows from Corollary 23.7 that $I_{\pi,\psi}(f) = \overline{L(v_{P,f})} = P(v_{L,\tilde{f}})$ and that $J_{\pi,\psi}(f) = \overline{L(v_{L,f})} = L(v_{L,\tilde{f}})$. Hence

$$I_{\pi,\psi}(f) = P(v_{L,\tilde{f}}) = \frac{1}{L(\pi,1/2)}\int L(\pi(t(a))v_{L,\tilde{f}})d^*a.$$

It is easy to check that $\pi(h)v_{\lambda,\tilde{f}} = v_{\lambda,(\rho_l(h)f)^\sim}$ for every continuous functional λ on H_∞ and every $f \in C_c^\infty(G)$ and $h \in G$. (Here $(\rho_l(h)f)(g) = f(h^{-1}g)$.) Hence $\pi(t(a))v_{L,\tilde{f}} = v_{L,(\rho_l(t(a))f)^\sim}$ and we get

$$I_{\pi,\psi}(f) = \frac{1}{L(\pi,1/2)}\int J_{\pi,\psi}(\rho_l(t(a))f)d^*a. \tag{3.6}$$

3.4. Invariance of the Bessel and relative Bessel distributions. For x in G we let $\rho_r(x)$ and $\rho_l(x)$ be as in (2.6) and (2.7). If $n \in N$ then it is easy to see that $v_{\rho_r(n)(f),L} = \pi(n)v_{f,L}$ and that $v_{\rho_l(n)(f),L} = \psi(n)v_{f,L}$. It follows that

$$\begin{aligned}(\rho_r(n)(J_{\pi,\psi}))(f) &= J_{\pi,\psi}(\rho_r(n^{-1})(f)) = \psi(n)J_{\pi,\psi}(f),\\ (\rho_l(n)(J_{\pi,\psi}))(f) &= J_{\pi,\psi}(\rho_l(n^{-1})(f)) = \psi(n)J_{\pi,\psi}(f).\end{aligned} \tag{3.7}$$

Let $a \in A$. It is easy to see that $v_{\rho_l(a)(f),P} = v_{f,P}$. Hence we have

$$(3.8)\qquad \begin{aligned} (\rho_r(n)(I_{\pi,\psi}))(f) &= I_{\pi,\psi}(\rho_r(n^{-1})(f)) = \psi(n)I_{\pi,\psi}(f), \\ (\rho_l(a)(I_{\pi,\psi}))(f) &= I_{\pi,\psi}(\rho_l(a^{-1})(f)) = I_{\pi,\psi}(f). \end{aligned}$$

3.5. BESSEL AND RELATIVE BESSEL FUNCTIONS. Let (π, H), $J_{\pi,\psi}$ and $I_{\pi,\psi}$ be as above. We can associate to $J_{\pi,\psi}$ and $I_{\pi,\psi}$ functions $j_{\pi,\psi}$ and $i_{\pi,\psi}$ as follows. Let $N^* = N - \{e\}$. Let $U_1 = NAw_0N$ and $U_2 = AN^*w_0N$.

THEOREM 3.2 ([2]): *There exists a real analytic function $j_{\pi,\psi}\colon U_1 \to \mathbf{C}$ such that*

$$(3.9)\qquad J_{\pi,\psi}(f) = \int_{U_1} f(g) j_{\pi,\psi}(g)dg, \quad f \in C_c^\infty(U_1).$$

Moreover, the Bessel function $j_{\pi,\psi}$ satisfies

$$j_{\pi,\psi}(n_1 g n_2) = \psi(n_1)\psi(n_2) j_{\pi,\psi}(g)$$

for every $n_1, n_2 \in N$ and $g \in U_1$.

Remark 3.3: The above theorem is proved in [2] in the generality of quasi-split reductive groups using a differential equations approach following Harish-Chandra's approach to the character function. We shall not need the above theorem in the sequel. We will define $j_{\pi,\psi}$ independently of $J_{\pi,\psi}$ using a kernel formula and then show that our new $j_{\pi,\psi}$ satisfies (3.9) for every $f \in C_c^\infty(G)$. Hence we can conclude that the two definitions of $j_{\pi,\psi}$ are the same.

The distribution $I_{\pi,\psi}$ satisfies the invariance relations (3.8) and is an eigendistribution for the center of the universal enveloping algebra of G. (See [17], p. 184.) We will now show that such distributions are given by functions when restricted to U_2.

Let $\square = 1/2H^2 + 1/2C^2 + XY + YX = 1/2H^2 + H + 1/2C^2 + 2YX$ be the Casimir element in the universal enveloping algebra of $\mathfrak{g}$. (See Section 2 for the definition of these elements.) Since $\square$ is in the center of the enveloping algebra of $\mathfrak{g}$, it follows that there exists a scalar $\alpha_\pi \in \mathbf{C}$ such that

$$\square I_{\pi,\psi} = \alpha_\pi I_{\pi,\psi}.$$

THEOREM 3.4: *Let I be a distribution on G satisfying:*

(1) $\rho_l(a)I = I, a \in A$.

(2) $\rho_r(n)I = \psi(n)I, n \in N$.

(3) $\Box I = \alpha I$ *for some* $\alpha \in \mathbf{C}$.

Then there exists a real analytic function $i\colon U_2 \to \mathbf{C}$ *such that*

$$I(f) = \int_{U_2} f(g)i(g)dg, \quad f \in C_c^\infty(U_2).$$

Moreover, the function i *satisfies*

$$i(agn) = \psi(n)i(g) \tag{3.10}$$

for every $a \in A$, $n \in N$ *and* $g \in U_2$. *If we set* $y(x) = i(n(x)w)$, $x \neq 0$, *and* $\psi(n(x)) = e^{2\pi i\lambda x}$ *then* y *satisfies the differential equation*

$$x^2y'' + (2x + 2\pi i\lambda)y' - \alpha y = 0. \tag{3.11}$$

Proof: We restrict I to the open set $U_2 = AN^*w_0N$. The mapping

$$(a, n_1, n_2) \mapsto an_1w_0n_2$$

is a submersion from $A \times N^* \times N$ to U_2. It follows that there exists an onto mapping from $C_c^\infty(A \times N^* \times N)$ to $C_c^\infty(U_2)$ which we now describe. Let $\beta \in C_c^\infty(A)$, $\gamma \in C_c^\infty(N^*)$ and $\delta \in C_c^\infty(N)$. We define $f_{\beta,\gamma,\delta} \in C_c^\infty(U_2)$ by

$$f_{\beta,\gamma,\delta}(an_1wn_2) = \beta(a)\gamma(n_1)\delta(n_2).$$

We set $\gamma(x) = \gamma(n(x))$. The following properties of this mapping are easy to verify:

$$\begin{aligned} &\mathrm{Ad}(\mathrm{n}_2)(\mathrm{H})\mathrm{f}_{\beta,\gamma,\delta} = \mathrm{f}_{-\mathrm{H}\beta,\gamma,\delta} + \mathrm{f}_{\beta,2\mathrm{x}\mathrm{X}\gamma,\delta}, \\ &\mathrm{Ad}(\mathrm{n}_2)(\mathrm{C})\mathrm{f}_{\beta,\gamma,\delta} = \mathrm{f}_{\mathrm{C}\beta,\gamma,\delta}, \\ &\mathrm{Ad}(\mathrm{n}_2)(\mathrm{YX})\mathrm{f}_{\beta,\gamma,\delta} = \mathrm{f}_{\beta,\mathrm{X}\gamma,\mathrm{X}\delta}. \end{aligned} \tag{3.12}$$

Since $\mathrm{Ad}(\mathrm{n}_2)(\Box) = \Box$, it follows from (3.12) that

$$\begin{aligned} \Box f_{\alpha,\beta,\gamma,\delta} = &f_{1/2H^2\beta,\gamma,\delta} \\ &+ f_{-H\beta,2xX\gamma,\delta} \\ &+ f_{\beta,2xXxX\gamma,\delta} \\ &+ f_{-H\beta,\gamma,\delta} \\ &+ f_{\beta,2xX\gamma,\delta} \\ &+ f_{1/2C^2\beta,\gamma,\delta} \\ &+ f_{\beta,2X\gamma,X\delta}. \end{aligned} \tag{3.13}$$

By the invariance assumptions (1) and (2) on I, there exists a distribution Y_I on N^* such that

$$I(f_{\beta,\gamma,\delta}) = Y_I(\gamma)\int_A \beta(a)d^*a\int \delta(n(y))\psi(n(y))dy \tag{3.14}$$

for every β, γ, δ as above. It follows from (3), from (3.13) and from (3.14) that Y_I satisfies

$$Y_I((2xXxX + 2xX - 4\pi i\lambda X)\gamma) = \alpha Y_I(\gamma), \quad \gamma \in C_c^\infty(N^*). \tag{3.15}$$

We define the differential operator D by

$$D = 2xXxX + 2xX - 4\pi i\lambda X - \alpha = 2x^2\frac{d^2}{dx^2} + 4x\frac{d}{dx} - 4\pi i\lambda\frac{d}{dx} - \alpha. \tag{3.16}$$

We identify N^* with $\mathbf{R}^*$. By (3.15) we have $Y_I(D\gamma) = 0$ for all $\gamma \in C_c^\infty(\mathbf{R}^*)$. Since D is an elliptic differential operator it follows that there exists an analytic function $y(x)$, $x \neq 0$ such that

$$Y_I(\gamma) = \int_{\mathbf{R}^*} \gamma(x)y(x)dx \tag{3.17}$$

and such that $y(x)$ is a solution for the differential equation $D^t y = 0$ where

$$D^t = 2XxXx - 2Xx + 4\pi i\lambda - \alpha = 2x^2\frac{d^2}{dx^2} + 4x\frac{d}{dx} + 4\pi i\lambda\frac{d}{dx} - \alpha. \tag{3.18}$$

Let da be the standard Haar measure on A. For $f \in C_c^\infty(U_2)$ define

$$O_f(x) = \int f(an(x)wn(y))\psi_\lambda(y)dady. \tag{3.19}$$

Then it follows from (3.14) and (3.17) that

$$I(f) = Y_I(O_f) = \int_{\mathbf{R}} O_f(x)y(x)dx. \tag{3.20}$$

Let $dg = dadxdy$ be a Haar measure on G restricted to U_2, the set of elements of the form $an(x)wn(y)$ for $a \in A$, $x, y \in \mathbf{R}$, $x \neq 0$. We define the function i on U_2 by

$$i(an(x)wn(y)) = y(x)\psi_\lambda(y).$$

Then it follows from (3.20) that for every $f \in C_c^\infty(U_2)$ we have

$$\begin{aligned} I(f) &= \int f(an(x)wn(y))i(an(x)wn(y))dadydx \\ &= \int_{U_2} f(g)i(g)dg \\ &= \int_G f(g)i(g)dg. \qquad \blacksquare \end{aligned}$$

For every distribution I satisfying conditions (1), (2) and (3) of Theorem 3.4 we let i_I be the function on U_2 which determines I on U_2. If $I = I_{\pi,\psi}$ is a relative Bessel distribution attached to a unitary representation π as above, then we set $i_{\pi,\psi} = i_{I_{\pi,\psi}}$.

COROLLARY 3.5: *Let I be a distribution on G satisfying (1), (2) and (3) of Theorem 3.4. Let $y(x) = i_I(n(x)w)$. Let ν be defined by the equation $-\alpha = 1/4 - \nu^2$. Then there exist constants c_1, c_2, c_3, c_4 such that*

$$y(x) = c_1|x|^{-1/2}e^{i\pi\lambda/x}J_\nu(\pi|\lambda|/x) + c_2|x|^{-1/2}e^{i\pi\lambda/x}J_{-\nu}(\pi|\lambda|/x) \tag{3.20}$$

when $x > 0$ and

$$y(x) = c_3|x|^{-1/2}e^{\frac{i\pi\lambda}{x}}J_\nu(\pi|\lambda|/|x|) + c_4|x|^{-1/2}e^{i\pi\lambda/x}J_{-\nu}(\pi|\lambda|/|x|) \tag{3.21}$$

when $x < 0$.

Proof: We need to solve the differential equation (3.11). Let $v(x) = y(x^{-1})$. Then v satisfies the differential equation

$$v'' - 2\pi i\lambda v' - \frac{\alpha}{x^2}v = 0. \tag{3.22}$$

It is easy to check that the functions $v_1(x) = |x|^{1/2}e^{i\pi\lambda x}J_\nu(\pi|\lambda x|)$ and $v_2(x) = |x|^{1/2}e^{i\pi\lambda x}J_{-\nu}(|\lambda x|)$ are independent solutions for (3.22) when ν is not an integer and that $|x|^{1/2}e^{i\pi\lambda x}J_\nu(|\lambda x|)$ and $|x|^{1/2}e^{i\pi\lambda x}Y_\nu(|\lambda x|)$ are independent solutions when ν is an integer. ∎

We can use the asymptotics of the classical Bessel functions J_ν and Y_ν (see [14], 5.11) to get:

COROLLARY 3.6: *Let I be a distribution as above and i_I be the function that gives I on U_2. There exists a positive constant E such that $|i_I(n(x)w_0)| < E$ for small x.*

We will want to prove a full regularity theorem for $J_{\pi,\psi}$ and $I_{\pi,\psi}$. That is, we will later prove that $J_{\pi,\psi}$ and $I_{\pi,\psi}$ are given by $j_{\pi,\psi}$ and $i_{\pi,\psi}$, respectively, on the full group and not only on U_1 and U_2. We know of two methods of doing that. The first is to prove a general regularity result for distributions satisfying the conditions of Theorem 3.4. We shall do that in the next two sections. We could also prove a general regularity result for distributions on G satisfying the invariance conditions of $J_{\pi,\psi}$. Instead we will use the classification of irreducible unitary representations and a kernel formula that will give a different definition of $j_{\pi,\psi}$ and show using this kernel formula that the distribution is given by a function on the full group. The advantage of the second approach is that we also get an explicit formula for $j_{\pi,\psi}$. Using this explicit formula for $j_{\pi,\psi}$ we will compute $i_{\pi,\psi}$.

3.6. ORBITAL INTEGRALS. In our discussion of the distributions $J_{\pi,\psi}$ and $I_{\pi,\psi}$, certain orbital integrals appeared naturally. It is worthwhile to define them again. Let $f \in C_c^\infty(U_2)$. By Theorem 3.2 there exists a function $j_{\pi,\psi}$ such that

$$\begin{aligned} J_{\pi,\psi}(f) &= \int_G f(g) j_{\pi,\psi}(g) dg \\ &= \int f(n_1 z s(b) w n_2) j_{\pi,\psi}(n_1 z s(b) w n_2) dn_1 dz dn_2 |b|^{-2} d^* b \\ &\quad + \int f(n_1 z s(b) w_0 n_2) j_{\pi,\psi}(n_1 z s(b) w_0 n_2) dn_1 dz dn_2 |b|^{-2} d^* b \\ &= \int_{\mathbf{R}^*} j_{\pi,\psi}(s(b) w) \left(\int f(n_1 z s(b) w n_2) \psi(n_1) \psi(n_2) dn_1 dz dn_2 \right) |b|^{-2} d^* b \\ &\quad + \int_{\mathbf{R}^*} j_{\pi,\psi}(s(b) w_0) \left(\int f(n_1 z s(b) w_0 n_2) \psi(n_1) \psi(n_2) dn_1 dz dn_2 \right) |b|^{-2} d^* b. \end{aligned}$$

Hence it is natural to define the (N, N) orbital integral

$$O_{f,\psi}^{N,N,Z}(g) = \int_{N \times N \times Z} f(n_1 z g n_2) \psi(n_1) \psi(n_2) dn_1 dz dn_2. \tag{3.23}$$

The distribution $I_{\pi,\psi}$ gives rise (see (3.20)) to the (A, N) orbital integral

$$O_{f,\psi}^{A,N}(g) = \int_{A \times N} f(a g w_0 n_1) \psi(n_1) da dn_1. \tag{3.24}$$

These orbital integrals were studied by Jacquet. In particular, it is proved in [11] that the integral in (3.23) converges absolutely for every $f \in C_c^\infty(G)$ and every $g \in U_2$ and that the integral in (3.24) converges absolutely for every $f \in C_c^\infty(G)$ and $g \in U_2$. Let $O_f(x) = O_{f,\psi}^{A,N}(n(x) w_0)$. Then for $f \in C_c^\infty(U_2)$ we have (see (3.20))

$$I_{\pi,\psi}(f) = \int_{\mathbf{R}} O_f(x) i_{\pi,\psi}(n(x) w_0) dx \tag{3.25}$$

and

$$\begin{aligned} I_{\pi,\psi}((\Box - \alpha_\pi) f) &= \int_{\mathbf{R}} D(O_f)(x) i_{\pi,\psi}(n(x) w_0) dx \\ &= \int_{\mathbf{R}} O_f(x) (D^t y)(x) dx = 0. \end{aligned} \tag{3.26}$$

Here $y(x) = i_{\pi,\psi}(n(x)w_0)$ and we are using the fact that

$$O_{(\square-\alpha_\pi)f}(x) = D(O_f)(x). \tag{3.27}$$

Here the differential operator D is defined in (3.16) and the above equation follows from the proof of Theorem 3.4 or from a direct calculation. Moreover, it is also easy to see that (3.27) is true for all $f \in C_c^\infty(G)$ and not just for $f \in C_c^\infty(U_2)$. We shall need this fact later in the sequel.

4. Orbital integrals

In this section we study the orbital integrals which were defined in (3.23) and (3.24). We are interested in bounds and explicit asymptotic expansions for these integrals. While it is possible to obtain explicit asymptotic expansions for both integrals, we will only need these expansions for the orbital integrals in (3.24). We start with some easy bounds for the orbital integrals in (3.23).

4.1. (N, N) ORBITAL INTEGRALS. Define

$$J_1(a,c,f) = \int f[n(x)z(c)s(a)w_0n(y)]dxdy,$$
$$J_2(a,c,f) = \int f[n(x)z(c)s(a)wn(y)]dxdy.$$

Notice that the integrals take place on two disjoint open sets whose union is the open cell. These two sets come from the two connected components of G.

PROPOSITION 4.1: *Let $f \in C_c^\infty(G)$. Then*

(a) *$J_i(a,c,f)$ converges absolutely for every $a, c \in \mathbf{R}^*$.*
(b) *$J_i(a,c,f)$ as a function of c is compactly supported in $\mathbf{R}^*$ independent of a.*
(c) *$J_i(a,c,f) = 0$ when $|a|$ is small (independent of c).*
(d) *$|J_i(a,c,f)| = O(|a|^{1+\epsilon})$ for every $\epsilon > 0$ when $|a|$ is large, independent of c.*

Proof: We shall prove everything for J_1. The proof for J_2 is the same. We first prove (b). f is compactly supported in G, hence the support of f is in a set of G on which the determinant is bounded. Since $|\det(n(x)z(c)s(a)w_0n(y)| = |c|^2$ we immediately get (b). Now, to prove (a), write

$$n(x)z(c)s(a)w_0n(y) = \begin{pmatrix} \frac{xc}{a} & \frac{xcy}{a} + ac \\ \frac{c}{a} & \frac{cy}{a} \end{pmatrix}.$$

Since c and a are fixed and nonzero, and since f is compactly supported on the $(1,1)$ entry and on the $(2,2)$ entry, it follows that the integration takes place on a bounded set in (x,y). Hence we have (a). By the above argument the integrand is identically zero for small values of c, hence by looking at the $(2,1)$ entry of the above matrix we see that the integrand vanishes for small values of a. Hence we have proved (c). Now we extend f to $\mathbf{R}^4$ by setting it to be zero outside of G. We make a change of variables $xca^{-1} \to x$ and $yca^{-1} \to y$. We have

$$J_1(a,c,f) = |a/c|^2 \int f \begin{pmatrix} x & \frac{xya}{c} + ac \\ \frac{c}{a} & y \end{pmatrix} dxdy.$$

Since f is bounded, we can bound the above integral by a constant times the area of a region of the form $\{x, y : |x| \le D_1, |y| \le D_2, |xya + ac^2| \le D_3\}$ for some constants D_1, D_2, D_3. It is easy to see that this area is of the order of $|a|^{-1}\log(|a|)$, hence we get our result. ∎

Corollary 4.2: *Let $f \in C_c^\infty(G)$ and let*

$$(4.1)\qquad \begin{aligned} O_1(a,f) &= \int |f[n(x)z(c)s(a)w_0n(y)]| dxdyd^*c, \\ O_2(a,f) &= \int |f[n(x)z(c)s(a)wn(y)]| dxdyd^*c. \end{aligned}$$

Then $O_i(a,f) = 0$ for $|a|$ small and $O_i(a,f) = O(|a|^{1+\epsilon})$ for $|a|$ large, $i = 1,2$ and $\epsilon > 0$.

Proof: There exists a positive function $\tilde{f} \in C_c^\infty(G)$ such that $\tilde{f} \ge |f|$. Applying Proposition 4.1 parts (b), (c) and (d) to $O_i(a,\tilde{f})$ gives the result. ∎

Taking absolute values inside the integral defining $O_{f,\psi}^{N,N,Z}$ we get:

Corollary 4.3: *$|O_{f,\psi}^{N,N,Z}(s(a)w)| = 0$ when $|a|$ is small and $|O_{f,\psi}^{N,N,Z}(s(a)w)| = O(|a|^{1+\epsilon})$. Similarly for $|O_{f,\psi}^{N,N,Z}(s(a)w_0)|$.*

4.2. (A,N) orbital integrals. Define

$$\begin{aligned} M_1(x,c,f) &= \int f[s(a)z(c)n(x)w_0n(y)]d^*ady, \\ M_2(x,c,f) &= \int f[s(a)z(c)n(x)wn(y)]d^*ady. \end{aligned}$$

PROPOSITION 4.4: *Fix $f \in C_c^\infty(G)$. Then*

(a) *$M_i(x,c,f)$ converges absolutely for every $x \neq 0$.*

(b) *$M_i(x,c,f) = 0$ for small and large values of $|c|$ independent of x.*

(c) *$M_i(x,c,f) = 0$ for large values of $|x|$ independent of c.*

(d) *$|M_i(x,c,f)| = O(|x|^\epsilon)$ for every $\epsilon > 0$ when $|x|$ is small (independent of c). The implied constant is also independent of c (but depends on ϵ).*

Proof: We shall prove the proposition for M_1. The proof is the same for M_2. We have

$$s(a)z(c)n(x)w_0n(y) = \begin{pmatrix} acx & ac(1+yx) \\ \frac{c}{a} & \frac{cy}{a} \end{pmatrix}.$$

Since the determinant of the above matrix is $-c^2$, it follows that the function vanishes for small and large values of $|c|$ independent of the other variables. Hence we have proved (b). Since the $(2,1)$ entry is $\frac{c}{a}$, it follows that the function vanishes for small values of $|a|$. Since the $(1,1)$ entry is cax and x is nonzero and fixed, we have that the function vanishes for large values of $|a|$ (depending on x). Finally, looking at the $(2,2)$ entry we get that the function vanishes for large values of $|y|$, hence the integral takes place over a compact set and is absolutely convergent. This proves (a).

For (c) we again look at the $(1,1)$ entry. Since the function vanishes for small values of $|c|$ and small values of $|a|$, it follows that the function vanishes for small values of $|x|$ independent of the other variables, hence we get (c).

For (d) we write

$$\begin{aligned} M_1(x,c,f) &= \int f\begin{pmatrix} acx & ac(1+yx) \\ \frac{c}{a} & \frac{cy}{a} \end{pmatrix} d^*ady \\ &= |c|^{-1}\int |a| f\begin{pmatrix} acx & ac(1+\frac{yax}{c}) \\ \frac{c}{a} & y \end{pmatrix} d^*ady \\ &= |x|^{-1}|c|^{-2}\int f\begin{pmatrix} a & \frac{a}{x}(1+\frac{ya}{c^2}) \\ \frac{c}{a} & y \end{pmatrix} dady. \end{aligned}$$

Now we can argue using volume considerations as in the proof of Proposition 4.1. ∎

We define

$$(4.2)\qquad \begin{aligned} \tilde{M}_1(x,f) &= \int f[s(a)z(c)n(x)w_0n(y)]d^*ady d^*c, \\ \tilde{M}_2(x,f) &= \int f[s(a)z(c)n(x)wn(y)]d^*ady d^*c. \end{aligned}$$

COROLLARY 4.5: (a) $\tilde{M}_i(x,f) = 0$ *when* $|x|$ *is large.*
(b) $|\tilde{M}_i(x,f)| = O(|x|^\epsilon)$ *for every* $\epsilon > 0$ *when* $|x|$ *is small.*

4.3. i IS LOCALLY INTEGRABLE.

PROPOSITION 4.6: *Let* I *be a distribution satisfying the conditions of Theorem 3.4 and let* $i = i_I$ *be the function that gives* I *on the set* U_2. *Then* i *is locally integrable on* G.

Proof: We need to show that

$$\int_G |i(g)|f(g)dg < \infty$$

for all positive $f \in C_c^\infty(G)$. We can replace G in the above integration with the set $Bw_0B - Aw_0N$ and restrict the Haar measure dg to this set. We have that $Bw_0B - Aw_0N$ is a disjoint union of the open sets $A_SZNw_0N - A_SZw_0N$ and $A_SZNwN - A_SZwN$. (See Section 2 for the definition of A_S.) Hence, up to a normalization of the Haar measure dg we can assume that $dg = d^*ad^*cdxdy$ on elements of the form $s(a)z(c)n(x)w_0n(y)$ and similarly on elements of the form $s(a)z(c)n(x)wn(y)$. Hence

$$\begin{aligned}\int_G |i_(g)|f(g)|dg &= \int_{A_SZw_0N} |i(g)|f(g)|dg + \int_{A_SZwN} |j(g)|f(g)|dg \\ &= \int |i(s(a)z(c)w_0n(y))|f(s(a)z(c)n(x)w_0n(y))d^*adyd^*cdx \\ &\quad + \int |i(s(a)z(c)n(x)wn(y))|f(s(a)z(c)n(x)wn(y))d^*adyd^*cdx \\ &= \int |i(n(x)w_0)|\tilde{M}_1(x,f)dx + \int |i(n(x)w)|\tilde{M}_2(x,f)dx.\end{aligned}$$

Here $\tilde{M}_1$ and $\tilde{M}_2$ are defined in (4.2). By Corollary 4.5 we have that the integrands vanish when $|x|$ is large. When $|x|$ is small it follows from Corollary 4.5 and Corollary 3.6 the integrands are of the order of $|x|^\epsilon$ for every $\epsilon > 0$, hence we get that both integrals are finite. ∎

It follows from the above corollary that the function i which comes from the distribution I gives a distribution T_i on the full group given by

$$T_i(f) = \int_G i(g)f(g)dg. \tag{4.3}$$

From Theorem 3.4 we get that $I = T_i$ on the set U_2. We would like to show that the two distributions, I and T_i, are the same on all of G, that is $I = T_i$.

To do that we need to show that the distribution $I - T_i$ satisfies conditions (1), (2) and (3) of Theorem 3.4. While (1) and (2) are obvious, we will need some more information to prove (3) for the distribution T_i. In particular, we will need explicit information about the asymptotics of the (A, N) orbital integrals.

4.4. Explicit asymptotics of (A, N) orbital integrals. We shall replace the (A, N) orbital integral with an integral on a different space. Let

$$\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can identify the space $A \setminus G$ with the space

$$X = \{g^{-1}\tau g : g \in G\} = \{x \in G : x^2 = 1, \det(x) = -1\}.$$

X is a closed submanifold in G and the map $Ag \mapsto g^{-1}\tau g$ is a diffeomorphism. In particular, every smooth function $\tilde{f}$ on $A \setminus G$ can be identified with a smooth function $\phi_{\tilde{f}}$ on X by

$$\phi_{\tilde{f}}(g^{-1}\tau g) = \tilde{f}(g). \tag{4.4}$$

Let $f \in C_c^\infty(G)$ and let $\tilde{f}$ be a function on $A \setminus G$ defined by

$$\tilde{f}(g) = \int_A f(ag)da \tag{4.5}$$

where da is a fixed Haar measure on A. Then $\tilde{f}$ is smooth and compactly supported on $A \setminus G$, hence $\phi_{\tilde{f}}$ is smooth and compactly supported on X.

Let $\psi(n(x)) = e^{2\pi i x}$. We can write

$$\begin{aligned} O_{f,\psi}^{A,N}(n(x)w_0) &= \int_{A\times N} f(an(x)w_0n_1)\psi(n_1)dadn_1 \\ &= \int_N \tilde{f}(n(x)w_0n_1)\psi(n_1)dadn_1 \\ &= \int \phi_{\tilde{f}}(n_1^{-1}w_0n(-x)\tau n(x)w_0n_1)\psi(n_1)dadn_1. \end{aligned}$$

Since X is closed in G, it follows that the function $\phi_{\tilde{f}}$ is a restriction of a function $\phi \in C_c^\infty(G)$ to X. In matrix form we can write the last integral as

$$G_\phi(x) = \int_{\mathbf{R}} \phi \begin{pmatrix} 2xy+1 & 2(xy^2+y) \\ -2x & -2xy-1 \end{pmatrix} e^{2\pi i y}dy. \tag{4.6}$$

Remark 4.7: We have chosen the character ψ to be $\psi(n(x)) = e^{2\pi i x}$. Similar arguments will work for the general case $\psi(n(x)) = e^{2\pi i\lambda x}$, but we prefer to simplify the notation by choosing $\lambda = 1$.

It follows from the above discussion that the asymptotics of $O^{A,N}_{f,\psi}(n(x)w_0)$ are the same as of $G_\phi(x)$ for the appropriate f and ϕ. Since $-2x$ appears in the $(2,1)$ entry and since ϕ is compactly supported on G, it follows that $G_\phi(x) = 0$ when $|x|$ is large. Hence it is enough to consider the case when $|x|$ is small. We will assume that $x > 0$. The case $x < 0$ can be treated similarly. We let $\beta = x^{-1}$ and we study the asymptotics of $F_\phi(\beta) = G_\phi(x) = G_\phi(\beta^{-1})$ when $\beta \to \infty$. We have

$$\begin{aligned} F_\phi(\beta) &= \int_{\mathbf{R}} \phi\begin{pmatrix} 2\beta^{-1}y+1 & 2(\beta^{-1}y^2+y) \\ -2\beta^{-1} & -2\beta^{-1}y-1 \end{pmatrix} e^{2\pi i y}dy \\ &= \beta\int_{\mathbf{R}} \phi\begin{pmatrix} 2y+1 & 2\beta(y^2+y) \\ -2\beta^{-1} & -2y-1 \end{pmatrix} e^{2\pi i\beta y}dy \\ &= \beta\int_{\mathbf{R}^2} \phi^{1,2}\begin{pmatrix} 2y+1 & z \\ -2\beta^{-1} & -2y-1 \end{pmatrix} e^{2\pi i\beta y}e^{4\pi i\beta z(y^2+y)}dydz. \end{aligned} \tag{4.7}$$

Here $\phi^{1,2}$ is the Fourier transform of ϕ in the $(1,2)$ component, that is,

$$\phi^{1,2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \int_{\mathbf{R}} \phi\begin{pmatrix} a & x \\ c & d \end{pmatrix} e^{2\pi i b x}dx$$

and we have used the Fourier inversion formula to get the last equation. For the first step we have used the change of variables $y \mapsto \beta y$.

PROPOSITION 4.8: *Let $F_\phi(\beta)$ be the function defined above and let α_1 be any smooth function on $(0,\infty)$ such that α vanishes around 0 and such that $\alpha(x) = 1$ for large x. Let*

$$c(\phi) = \frac{1}{2}\int_{-\infty}^{\infty} \phi\begin{pmatrix} -1 & x \\ 0 & 1 \end{pmatrix} e^{\pi i x}dx.$$

For $\beta > 0$ we write

$$F_\phi(\beta) = c(\phi)e^{-2\pi i\beta}\alpha_1(\beta) + \alpha_2(\beta)$$

where α_2 is a smooth function on $(0,\infty)$ which depends on ϕ and α_1. Then α_2 vanishes around 0 and

$$\alpha_2(\beta) = O(1), \alpha_2'(\beta) = O(\beta^{-1})$$

at ∞.

Proof: The $(2,1)$ entry of the matrix in the integral (4.7) is $-2\beta^{-1}$. When $\beta \to \infty$ this entry goes to zero. By using a Taylor series for $\phi^{1,2}$ at the $(2,1)$

variable we can see that the main term of the asymptotics of $F_\phi(\beta)$ comes from the integral

$$\tilde{F}_\phi(\beta) = \int_{\mathbf{R}} \phi^{1,2} \begin{pmatrix} 2y+1 & z \\ 0 & -2y-1 \end{pmatrix} e^{2\pi i \beta y} e^{4\pi i \beta z (y^2+y)} dydz.$$

Now we use the stationary phase method (see [19]) to obtain the asymptotics of the above integral. The phase function is

$$\beta(y,z) = 2\pi(y + 2z(y^2+y))$$

and it has two critical points: $(-1/2, 0)$ and $(1/2, -1)$. It is easy to check that the Hessian at these points in nonzero, hence β is a Morse function. Now it follows from ([19], Theorem 2.9) that $\tilde{F}_\phi(\beta)$ has an asymptotic expansion at infinity of the form

$$(4.8) \quad \tilde{F}_\phi(\beta) = e^{2\pi i \beta}(c_1 + c_2\beta^{-1} + c_3\beta^{-2} + \cdots) + d_1 + d_2\beta^{-1} + d_3\beta^{-2} + \cdots.$$

Here $c_1 = c(\phi)$ as defined above. Notice that this infinite sum does not converge. We use here the convention of ([19], 2.2).

The next terms in the Taylor expansion will give an integral which will have the same form as (4.8) but with smaller terms. Hence $F_\phi(\beta)$ will also have an asymptotic expansion of the form (4.8) with $c_1 = c(\phi)$. By taking the derivatives of $F_\phi(\beta)$ in the integral (4.7), it is easy to see that all derivatives of $F_\phi(\beta)$ have asymptotic expansions of the type (4.8) and that they can be obtained by taking the derivative of the asymptotic expansion for $F_\phi(\beta)$. Notice that for our result we only need the first derivative, the first term of the asymptotic expansion and the first term and the remainder in the Taylor expansion. ∎

Remark 4.9: Notice that the function $\phi^{1,2}$ is not compactly supported in the $(1,2)$ entry. However, it is a Schwartz function in that variable and in all variables. To overcome this we look at the integral (4.7) when $|z|$ is large. We make the change of variables $u = z$ and $v = y + 2z(y^2+y)$. It is easy to see that this transformation is nonsingular when $|z|$ is large and that the resulting integral gives a rapidly decreasing function in β. Hence, by splitting our integral using a partition of unity into the two sets where $|z|$ is large and the complement, we get one integral that gives a rapidly decreasing function in β and another which involves a compactly supported function in the variables y and z, hence the expansion (4.8) is still valid.

Recall that $G_\phi(x) = F_\phi(x^{-1})$ is defined in (4.6).

COROLLARY 4.10: *Let α_1 be any smooth function on $(0,\infty)$ such that α vanishes around ∞ and such that $\alpha(x)=1$ in a neighborhood of* 0. *Write*

$$G_\phi(x) = c(\phi)e^{-2\pi i/x}\alpha_1(x) + \alpha_2(x)$$

where α_2 is a smooth function on $(0,\infty)$ which depends on ϕ and α_1. Then α_2 vanishes around ∞ and for small x we have

$$\alpha_2(x) = O(1), \alpha_2'(x) = O(x^{-1}).$$

If $\phi = \phi_{\tilde f}$ as defined in (4.4) and (4.5), where $f \in C_c^\infty(G)$ and $\tilde f \in C_c^\infty(A\backslash G)$, then it is easy to see that

$$c(\phi_{\tilde f}) = \int_{A\times N} f(an)\psi(n)dn = Q(f). \tag{4.9}$$

COROLLARY 4.11: *Let $f \in C_c^\infty(G)$. Let α_1 be any smooth function on $(0,\infty)$ such that α vanishes around ∞ and such that $\alpha(x)=1$ in a neighborhood of* 0. *For $x>0$, write*

$$O_{f,\psi}^{A,N}(n(x)w_0) = Q(f)e^{2\pi i/x}\alpha_1(x) + \alpha_2(x)$$

where α_2 is a smooth function on $(0,\infty)$ which depends on ϕ and α_1. Then α_2 vanishes around ∞ and

$$\alpha_2(x) = O(1), \alpha_2'(x) = O(x^{-1}).$$

Similarly for $x<0$ we get

PROPOSITION 4.12: *Let $f \in C_c^\infty(G)$. Let α_1 be any smooth function on $(-\infty,0)$ such that α vanishes around $-\infty$ and such that $\alpha(x)=1$ in a neighborhood of* 0. *For $x<0$, write*

$$O_{f,\psi}^{A,N}(n(x)w_0) = Q(f)e^{-2\pi i/x}\alpha_1(x) + \alpha_2(x)$$

where α_2 is a smooth function on $(-\infty,0)$ which depends on ϕ and α_1. Then α_2 vanishes around $-\infty$ and

$$\alpha_2(x) = O(1), \alpha_2'(x) = O(x^{-1}).$$

5. Regularity of the relative Bessel distributions

In this section we will show that the relative Bessel distributions $I_{\pi,\psi}$ (see (3.5)) are given by the relative Bessel functions on the full group. At this point, after the results of Section 4, we are given two distributions, I and T_i, which agree on an open set. We would like to show that they are equal everywhere. The first distribution satisfies a differential equation coming from the Casimir element. We need to obtain some information about the action of the Casimir operator on the second distribution. This is the content of our first Theorem.

Let I be a distribution satisfying conditions (1), (2) and (3) of Theorem 3.4 and let i be the function which gives I on the set U_2. Let $y(x) = i(n(x)w_0)$. Then y is given by (3.20) and (3.21) and i is given on U_2 by (3.10). By the proof of Proposition 4.6 it follows that any function i which is defined on U_2 as above is locally integrable and gives a distribution T_i on G defined by (4.3).

THEOREM 5.1: *Let i be a function on U_2 defined as above. That is, $y(x) = i(n(x)w_0)$ is defined by (3.20) and (3.21) for some constants c_1, c_2, c_3, c_4 and i is extended to U_2 by (3.10). Let T_i be the distribution defined by (4.3). Then there exists $\beta \in \mathbf{C}$ such that*

$$(\Box - \alpha)T_i = \beta Q$$

where Q is a distribution coming from the Borel subgroup B and Q is given by (4.9).

Proof: We will assume that $\psi(n(x)) = e^{2\pi i x}$. Hence $\lambda = 1$ in (3.11) and (3.20), (3.21). The general case, $\psi(n(x)) = e^{2\pi i\lambda x}$, is similar. By (3.25) we have that

$$T_i(f) = \int_G f(g)i(g)dg = \int_{\mathbf{R}} O_f(x)i(n(x)w_0)dx = \int_{\mathbf{R}} O_f(x)y(x)dx$$

and by (3.27)

$$(\Box - \alpha)T_i(f) = \int_{\mathbf{R}} D(O_f)(x)y(x)dx.$$

Notice that the function $y(x)$ satisfies the differential equation $D^t(y) = 0$ for $x \neq 0$. We divide the above integral into two parts: from 0 to ∞ and from $-\infty$ to 0. For each part we apply integration by parts. To compute the integration by parts we use the expression for $O_f(x)$ given by Corollary 4.11 and by Proposition 4.12. We also need to use the asymptotic expansions for the classical Bessel functions (see, for example, [14], 5.11). ∎

PROPOSITION 5.2: *Let T be a distribution on G satisfying:*

(1) $\rho_r(n)I = \psi(n)I, n \in N$.

(2) *There exists $\alpha \in \mathbf{C}$ such that $(\Box - \alpha)T = Q$, where Q is a distribution which is defined on B (and extended to G).*

(3) *T is supported on $G - U_2$, that is T is supported on the disjoint union of B and $w_0 B$.*

Then $T = 0$.

Proof: The proof is similar to the proof of Proposition 2.10 in [17]. We shall assume that $\psi(n(x)) = e^{2\pi i x}$. The general case is similar. Let X be as in Section 2. Then it follows from (1) that

$$X(T) = 2\pi i T$$

where we identify X with the left invariant differential operator defined in (2.8). We write $\Box = 2YX + 1/2H^2 + H + 1/2C^2$, hence (2) gives

$$4\pi i Y(T) = (-1/2H^2 - H - 1/2C^2 + \alpha)T + Q. \tag{5.1}$$

Let L be the one-dimensional Lie algebra spanned by Y which we view as a Lie algebra of left invariant vector fields. Let B be the Borel subgroup. Then L is transversal to B in the sense of ([17], (2.2) and (2.3)). Here we can take M to be G or to be $G - w_0 B$. We restrict our distribution T to the open set $M = G - w_0 B$. Then T on M is supported on B. Let $b \in B$. By ([17], Lemma 2.4), there exists a neighborhood of b such that

$$T = \sum Y^j T_j \tag{5.2}$$

on this neighborhood and T_j are distributions defined on B. Here the sum is finite, starts with $j = 0$ and the T_j are unique. We apply equation (5.1) to the above formula. Then we have

$$8\pi i \sum Y^{j+1} T_j = \sum (-H^2 - 2H - C^2 + 2\alpha) Y^j T_j + 2Q.$$

We now rewrite the left and the right side into expressions of the form

$$\sum Y^j Q_j$$

for distributions Q_j defined on B. If $T \neq 0$, then we get different expressions for the left hand side and the right hand side since the order of the highest nonzero term will be bigger in the left hand side. This can be seen directly by commuting Y with the H and C, or by Lemma 2.5 in [17]. It follows from the uniqueness that $T = 0$ in a neighborhood of b, hence $T = 0$ on M and T is supported on w_0B. We also get from (2) that $Q = 0$.

We now set $M = G$ and L as before is spanned by Y. We claim that L is transversal to w_0B in the sense of ([17], (2.2) and (2.3)). By left invariance it is enough to show that for all $p \in w_0B$, L is transversal to $p^{-1}w_0B$ at e. If $p = w_0b$ then $p^{-1}w_0B = b^{-1}B = B$, hence the claim is clear. Now the proof proceeds exactly as above. T satisfies (5.1) with $Q = 0$. In a neighborhood of p we can write T as in (5.2) where T_i are distributions defined on w_0B. Now, the same arguments will show that $T_i = 0$ for all i, hence $T = 0$ in a neighborhood of p, hence $T = 0$ everywhere. ∎

Our main theorem of the last three sections is the following.

THEOREM 5.3: *Let I be a distribution on G satisfying conditions (1), (2) and (3) of Theorem 3.4. That is, assume*

(1) $\rho_l(a)I = I, a \in A$.
(2) $\rho_r(n)I = \psi(n)I, n \in N$.
(3) $\Box I = \alpha I$ *for some* $\alpha \in \mathbf{C}$.

Then there exists a real analytic function i on U_2 which is locally integrable on G such that

$$I(f) = \int i(g)f(g)$$

for every $f \in C_c^\infty(G)$.

Proof: By Theorem 3.4 there exists a function $i(g)$ such that the above equality is true on U_2, that is, for $f \in C_c^\infty(U_2)$. By Proposition 4.6, i is locally integrable. Let T_i be the distribution given by i. (See (4.3).) Let $T = T_i - I$. Then T is supported on $G - U_2$ and, by (3) and by Theorem 5.1,

$$(\Box - \alpha)T = Q$$

where Q is a distribution defined on B. (Notice that we are replacing the distribution βQ in Theorem 5.1 with Q.) It follows from Proposition 5.2 that $T = 0$, that is $I = T_i$, which is the required conclusion. ∎

Remark 5.4: We have also proved that $Q = 0$, that is, $\beta = 0$ in Theorem 5.1. This will give a condition on the scalars c_1, c_2, c_3, c_4 that appear in the formulas for $i(g)$. We will not need these conditions in the sequel.

Remark 5.5: It is possible to prove a regularity result for the Bessel distributions in the same way, that is, let J be a distribution on G satisfying $\rho_l(n)J = \rho_r(n)J = \psi(n)J$ for every $n \in N$ and $\rho_r(z)J = \chi(z)J$ for some quasi-character χ on Z and every $z \in Z$. Assume also that there exists $\alpha \in C$ such that $\Box J = \alpha J$. Then we can prove that there exists a real analytic function j on Bw_0B which is locally integrable on G such that

$$J(f) = \int j(g)f(g)dg$$

for every $f \in C_c^\infty(G)$. We shall proceed in a different way and prove a weaker result which is sufficient for our purpose. Starting from a unitary representation π of G with a trivial central character, we shall use the results of Cogdell and Piatetski-Shapiro to give $j_{\pi,\psi}$ as a kernel function for the action of w_0 in the Kirillov model. After that we will use this Kernel formula to show that this function is one and the same as the Bessel function coming from the Bessel distribution $J_{\pi,\psi}$ and that it gives the distribution on the full group.

6. Bessel functions for $GL_2(\mathbf{R})$

In this section we recall the formula in [5] for the action of the Weyl element in the Kirillov or Whittaker model of an irreducible unitary representation with trivial central character of $GL_2(\mathbf{R})$. This action will be given by a kernel formula involving a classical Bessel function. We include the proof of this kernel formula in Appendix 2 in order to make this discussion self-contained. We mention that our method of computation and the proof of this kernel formula is different than the computation and proof mentioned in [5] which was communicated to us by Jim Cogdell. We thank Jim Cogdell for explaining to us this result. Our method of proof is similar to the method indicated in ([20], chapter VII, section 7). We introduce a convergence factor that does not appear in [20] in order to allow a change of order of integration. After finishing our original manuscript, we learned of a different approach for the proof using a Mellin transform which was obtained around the same time by Motohashi [15] in the case of the principal series.

We shall employ the same proof and computation for the double cover of $GL_2(\mathbf{R})$ to get a new kernel formula in the Kirillov model of unitary genuine representations.

Let π be an irreducible admissible representation of $G = GL_2(\mathbf{R})$ on a Hilbert space H with trivial central character. Let $\psi = \psi_\lambda$ be the character of N defined in Section 2. Let dx and d^*x be the measures defined in (2.1).

Fix a nonzero ψ Whittaker functional (see (2.4)) L on H_∞. For $v \in H_\infty$ we set $W_v(g) = L(\pi(g)v)$.

THEOREM 6.1 ([5] (see also Appendix 2)): *Fix π and L as above. There exists a function $j_1\colon \mathbf{R}^* \to \mathbf{C}$ such that*

$$W_v(t(y)w_0) = \int_{\mathbf{R}^*} j_1(yx)W_v(t(x))d^*x \tag{6.1}$$

for all $v \in H_\infty$ such that $W_v(t(x))$ has a high order of vanishing at $a = 0$.

Remark 6.2: The above integral converges absolutely. The exact order of vanishing that is needed will depend on π and will be given explicitly in the proof. When π is a unitary principal series or a unitary complementary series, we will show that every $v \in H_\infty$ satisfies the required condition, hence the formula is valid for every $v \in H_\infty$. For the discrete series we will show that all the K finite vectors satisfy the condition. We believe that the formula is valid for all $v \in H_\infty$ in the discrete series.

Notice that j_1 is denoted by $k_{\lambda,\eta,s}$ in Appendix 2.

We extend j_1 to Bw_0B by

$$j_{\pi,\psi}(n(r)z(c)t(a)w_0n(s)) = \psi(n(r))\psi(n(s))j_1(a).$$

The following formula is an easy consequence of (6.1) (see [1] for an argument).

THEOREM 6.3: *Assume $v \in H_\infty$ is as above. Then*

$$W_v(g) = \int_{\mathbf{R}^*} j_{\pi,\psi}(gt(a^{-1}))W_v(t(a))d^*a \tag{6.2}$$

for all $g \in Bw_0B$.

We shall now give the formulas in [5] for $j_{\pi,\psi}(t(x)w_0)$. We note that the formulas in ([5], p. 57) are for $j_{\pi,\psi}(t(x)w)$, hence their values for x will be our values for $-x$. Let $\mu(x) = |x|^s(\mathrm{sgn}(x))^\eta$ with $\mathrm{Re}(s) \geq 0$ be a quasi-character of $\mathbf{R}^*$. We denote by $\pi(\mu,\mu^{-1})$ the representation of G induced by the character (μ,μ^{-1}) from B if it is irreducible. (We use normalized induction.) If it is not irreducible then $\pi(\mu,\mu^{-1})$ will denote its unique irreducible subspace. The representation $\pi(\mu,\mu^{-1})$ is denoted by $\pi_{\eta,s}$ in Appendix 2.

THEOREM 6.4 ([5]): *Let $\psi = \psi_\lambda$ as defined in Section 2.*

(i) *If $\mu(x) = |x|^{d-1/2}\,\mathrm{sgn}(x)$ with $d \in \mathbf{N}$, then $\pi = \pi(\mu,\mu^{-1})$ is a holomorphic discrete series representation and we have*

$$j_{\pi,\psi}(t(x)w_0) = \begin{cases} (-1)^d 2|\lambda|^{1/2}\pi|x|^{1/2}J_{2d-1}(4\pi|\lambda|\sqrt{|x|}) & \text{if } x < 0; \\ 0 & \text{if } x > 0. \end{cases}$$

(ii) *If* $\mu(x) = |x|^{ir}(\mathrm{sgn}(x))^\eta$ *with* $r \in \mathbf{R}$ *and* $\eta \in \{0,1\}$, *then* $\pi = \pi(\mu, \mu^{-1})$ *is a principal series representation and we have*

$$j_{\pi,\psi}(t(x)w_0) = \begin{cases} 4(-1)^\eta |\lambda|^{1/2}|x|^{1/2}\cos(\pi i r)K_{2ir}(4\pi|\lambda|\sqrt{|x|}) & \text{if } x > 0; \\ |\lambda|^{1/2}|x|^{1/2}\frac{-\pi}{\sin(\pi i r)}(J_{2ir}(4\pi|\lambda|\sqrt{|x|}) - J_{-2ir}(4\pi|\lambda|\sqrt{|x|})) & \\ & \text{if } x < 0. \end{cases}$$

(iii) *If* $\mu(x) = |x|^r(\mathrm{sgn}(x))^\eta$ *with* $0 < r < 1/2$ *and* $\eta \in \{0,1\}$, *then* $\pi = \pi(\mu, \mu^{-1})$ *is a complementary series representation and we have*

$$j_{\pi,\psi}(t(x)w_0) = \begin{cases} 4(-1)^\eta |\lambda|^{1/2}|x|^{1/2}\cos(\pi r)K_{2r}(4\pi|\lambda|\sqrt{|x|}) & \text{if } x > 0; \\ |\lambda|^{1/2}|x|^{1/2}\frac{-\pi}{\sin(\pi r)}(J_{2r}(4\pi|\lambda|\sqrt{|x|}) - J_{-2r}(4\pi|\lambda|\sqrt{|x|})) & \\ & \text{if } x < 0. \end{cases}$$

Since the central character of π is trivial and since

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} = \begin{pmatrix} a^2 & \\ & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & \\ & a^{-1} \end{pmatrix},$$

we have

$$j_{\pi,\psi}(s(a)w_0) = j_{\pi,\psi}(t(a^2)w_0), \quad j_{\pi,\psi}(s(a)w) = j_{\pi,\psi}(t(-a^2)w_0).$$

Using the theory of classical Bessel functions [14] we get:

LEMMA 6.5: *Let* π *be a unitary irreducible representation of* G *with trivial central character. Then*

$$j_{\pi,\psi}(s(a)w_0) = O(|a|^{1/2}) \quad \text{and} \quad j_{\pi,\psi}(s(a)w) = O(|a|^{1/2})$$

when $|a|$ *is large.*

PROPOSITION 6.6: $j_\pi = j_{\pi,\psi}$ *is locally integrable on* G.

Proof: We need to show that

$$\int_G |j_\pi(g)f(g)|dg < \infty$$

for all $f \in C_c^\infty(G)$. We can replace G in the above integration with the set Bw_0B and restrict the Haar measure dg to this set. We have that Bw_0B is a disjoint union of the open sets NZA_Sw_0N and NZA_SwN. Hence we have

$$\int_G |j_\pi(g)f(g)|dg = \int_{NZA_Sw_0N} |j_\pi(g)f(g)|dg + \int_{NZA_SwN} |j_\pi(g)f(g)|dg.$$

Since

$$\int_{NZA_Sw_0N} |j_\pi(g)f(g)|dg$$
$$= \int |j_\pi(n(x)z(c)s(a)w_0n(y))f(n(x)z(c)s(a)w_0n(y))||a|^{-2}dxdyd^*c$$

and similarly for the second summand, we get that

$$\int_G |j_\pi(g)f(g)|dg = \int |j_\pi(s(a)w_0)|O_1(a,f)||a|^{-2}d^*a$$
$$+ \int |j_\pi(s(a)w)|O_2(a,f)||a|^{-2}d^*a.$$

Here O_1 and O_2 are defined in (4.1). By Corollary 4.2 we have that the integrands vanish when $|a|$ is small. It follows from Corollary 4.2 and Lemma 6.5 that the integrands are of order $|a|^{-1/2+\epsilon}$ when $|a|$ is large. Hence it is clear that both integrals are finite. ∎

7. Bessel distributions for $GL_2(\mathbf{R})$

In this section we show that the Bessel distributions on $GL_2(\mathbf{R})$ (or to be precise, the Bessel distributions attached to irreducible unitary representation of $GL_2(\mathbf{R})$ with trivial central character) are given by the Bessel functions defined in Section 6. The proof here is similar to the one given in [1] for the p-adic case.

Let $G = GL_2(\mathbf{R})$ and let (π, H) be an infinite-dimensional irreducible unitary representation of G with trivial central character. Let L be a ψ Whittaker functional on H_∞.

For $f \in C_c^\infty(G)$ we let $v_{f,L} \in H_\infty$ be the unique vector that satisfies

$$L(\pi(f)u) =< u, v_{f,L} >$$

for all $u \in H$. Here $<,>$ is the inner product defined in (3.1). (See Remark 3.1.) As in (3.4) we define the normalized Bessel distribution by

$$J_{\pi,\psi}(f) = \overline{L(v_{f,L})}.$$

Lemma 7.1: *Let $u \in H_\infty$. Then*

$$\int_{\mathbf{R}^*} J_{\pi,\psi}(\rho_r(t(a))f)W_u(t(a))d^*a = \int_G f(g)W_u(g)dg.$$

Proof: It is easy to see that $v_{\rho_r(t(a))f,L} = \pi(t(a))(v_{f,L})$. Hence

$$J_{\pi,\psi}(\rho_r(t(a))f) = \overline{W_{v_{f,L}}(t(a))}.$$

Thus, by (3.1),

$$\int_{\mathbf{R}^*} J_{\pi,\psi}(\rho_r(t(a))f)W_u(t(a))d^*a = < u, v_{f,L} > .$$

By definition

$$< u, v_{f,L} >= L(\pi(f)u) = \int_G f(g)L(\pi(g)u)dg = \int_G f(g)W_u(g)dg. \qquad \blacksquare$$

Define a distribution $\tilde{J}_{\pi,\psi}$ on $C_c^\infty(G)$ by

$$\tilde{J}_{\pi,\psi}(f) = \int_G f(g)j_{\pi,\psi}(g)dg.$$

LEMMA 7.2: *Let $u \in H_\infty$ and $f \in C_c^\infty(G)$. Assume that $W_u(t(a))$ has a high order of vanishing at $a = 0$ (say $W_u(t(a)) = O(|a|^2)$). Then*

$$\int \tilde{J}_{\pi,\psi}(\rho_r(t(a))f)W_u(t(a))d^*a = \int_G f(g)W_u(g)dg.$$

Proof:

$$\begin{aligned}\int \tilde{J}_{\pi,\psi}(\rho_r(t(a))f)W_u(t(a))d^*a =&\int_{\mathbf{R}^*} W_u(t(a))\left(\int_{Bw_0B} f(g(t(a)))j_{\pi,\psi}(g)dg\right)d^*a\\ =&\int_{\mathbf{R}^*} W_u(t(a))\left(\int_{Bw_0B} f(g)j_{\pi,\psi}(gt(a^{-1}))dg\right)d^*a\\ =&\int_{Bw_0B} f(g)\left(\int_{\mathbf{R}^*} j_{\pi,\psi}(gt(a^{-1}))W_u(t(a))d^*a\right)dg\\ =&\int_{Bw_0B} f(g)W_u(g)dg.\end{aligned}$$

Here we have obtained the last equality from (6.2). To justify the change of order above consider the integral

$$\begin{aligned}&\left|\int_{Bw_0B} f(g)j_{\pi,\psi}(gt(a^{-1}))dg\right|\\ &\leq \int |f(n_1zt(b)w_0n_2)||j_{\pi,\psi}(n_1zt(b)w_0n_2t(a^{-1}))|dn_1dzdn_2|b|^{-2}db\\ &= \int |j_{\pi,\psi}(t(b)t(a^{-1})w_0)|\left(\int |f(n_1zt(b)w_0n_2)|dn_1dzdn_2\right)|b|^{-2}db.\end{aligned}$$

By Corollary 4.2, the inner integral vanishes for small $|b|$ and is of the order $|b|^{1/2+\epsilon}$ when $|b|$ is large. By Theorem 6.4, $j_{\pi,\psi}(t(c)w_0) = O(|c|^{1/4})$ when $|c|$ is large and $j_{\pi,\psi}(t(c)w_0) = O(1)$ when $|c|$ is small, hence it follows that the integral gives a function of a which has an order of $|a|^{1/2}$ at ∞ and $|a|^{-1/4}$ at $a = 0$. Since $W_u(t(a))$ is rapidly decreasing at ∞ and has a large order of vanishing at $a = 0$, the integral converges absolutely and we can change the order. ∎

Corollary 7.3:

$$J_{\pi,\psi}(f) = \int_G f(g) j_{\pi,\psi}(g) dg.$$

Proof: By Lemma 7.1 and Lemma 7.2 we have that if $u \in H_\infty$ and $W_u(t(a))$ has a high order of vanishing at $a = 0$, then

$$\int J_{\pi,\psi}(\rho_r(t(a))f) W_u(t(a)) d^*a = \int \tilde{J}_{\pi,\psi}(\rho_r(t(a))f) W_u(t(a)) d^*a$$

for all $f \in C_c^\infty(G)$. If $u \in H_\infty$ is such that $W_u(t(a))$ has a high order of vanishing at $a = 0$, then $\pi(n)u$ has the same property for every $n \in N$, hence we can replace u with $\pi(n)u$ in the above equality. It follows from Lemma 2.1 that $J_{\pi,\psi}(\rho_r(t(a))f)W_u(t(a)) = \tilde{J}_{\pi,\psi}(\rho_r(t(a))f)W_u(t(a))$ for all $a \in \mathbf{R}^*$. It is well known (see, for example, [9]) that there exists $u \in H_\infty$ such that $W_u(t(a))$ has a high order of vanishing at $a = 0$ and such that $W(e) = W(t(1)) = 1$. Hence $J_{\pi,\psi}(f) = \tilde{J}_{\pi,\psi}(f)$. ∎

Remark 7.4: A similar proof works for general irreducible admissible representations of $GL_2(\mathbf{R})$. Since such a representation is not always unitary, we need to replace the invariant inner product in the definition of the Bessel distribution with the invariant pairing between the representation and its dual. For more details see [1].

8. Relative Bessel functions on $GL_2(\mathbf{R})$

In this section we compute the relative Bessel function of the infinite-dimensional irreducible unitary representations of G with trivial central character. In the heart of this section and of this paper is a computation of the Fourier transform of the Bessel functions obtained in Section 7. This computation is based on a Fourier transform of certain Bessel functions obtained by Hardy and Weber [6]. We carry out this computation in Appendix 3.

Let π be an infinite-dimensional irreducible unitary representation of G with trivial central character. Let $I_{\pi,\psi}$ and $J_{\pi,\psi}$ be the normalized Bessel and relative

Bessel distributions as defined in (3.4) and (3.5). By (3.6) we have

$$I_{\pi,\psi}(f)=\frac{1}{L(\pi,1/2)}\int J_{\pi,\psi}(\rho_l(t(b))f)d^*b. \tag{8.1}$$

In particular, since $J_{\pi,\psi}$ is given by $j_{\pi,\psi}$ on $U_2=Bw_0B-Aw_0N$ and $I_{\pi,\psi}$ is given by $i_{\pi,\psi}$ on U_2, we have that

$$\frac{1}{L(\pi,1/2)}\int_{\mathbf{R}^*}\left(\int_{U_2}f(t(b)g)j_{\pi,\psi}(g)dg\right)d^*b=\int_{U_2}f(g)i_{\pi,\psi}(g)dg \tag{8.2}$$

for all $f\in C_c^\infty(U_2)$. We shall use (8.2) to find a more explicit relation between $i_{\pi,\psi}$ and $j_{\pi,\psi}$ and consequently compute $i_{\pi,\psi}$.

LEMMA 8.1: *Let $f\colon U_2\to\mathbf{C}$ be defined by*

$$f(t(a)n(x)z(c)w_0n(y))=\phi_1(a)\phi_2(x)\phi_3(c)\phi_4(y) \tag{8.3}$$

where $\phi_1,\phi_2,\phi_3\in C_c^\infty(\mathbf{R}^*)$ and $\phi_4\in C_c^\infty(\mathbf{R})$. Then

$$\begin{aligned}&\int_{R^*}\int_G f(t(b)g)j_{\pi,\psi}(g)dgd^*b\\&=\int\phi_1(b)d^*b\int\phi_3(c)d^*c\int\phi_4(y)\psi(y)dy\int j_{\pi,\psi}(t(a)w_0)\hat{\phi}_2(a)d^*a\end{aligned}$$

where $\hat{\phi}_2$ is the λ-Fourier transform of ϕ_2 as defined in (2.2).

Proof: We let $dg=d^*adxd^*cdy$ be a Haar measure on the set of elements of the form $g=t(a)n(x)z(c)w_0n(y)$. Then

$$\int_{R^*}\int_G f(t(b)g)j_{\pi,\psi}(g)dgd^*b \tag{8.4}$$

$$\begin{aligned}&=\int\phi_1(ba)\phi_2(x)\phi_3(c)\phi_4(y)j_{\pi,\psi}(t(a)n(x)z(c)w_0n(y))d^*adxd^*cdy\\&=\int\phi_3(c)d^*c\int\phi_4(y)\psi(y)dy\left(\int\phi_2(x)\psi(ax)dx\int\phi_1(ba)j_{\pi,\psi}(t(a)w_0)d^*a\right)d^*b\\&=\int\phi_3(c)d^*c\int\phi_4(y)\psi(y)dy\int\phi_1(ba)j_{\pi,\psi}(t(a)w_0)\hat{\phi}_2(a)d^*ad^*b.\end{aligned}$$

We claim that the last integral converges absolutely. Hence we change the order of integration to get

$$\begin{aligned}\int\phi_1(ba)j_{\pi,\psi}(t(a)w_0)\hat{\phi}_2(a)d^*ad^*b&=\int\phi_1(ba)j_{\pi,\psi}(t(a)w_0)\hat{\phi}_2(a)d^*bd^*a\\&=\int\phi_1(b)j_{\pi,\psi}(t(a)w_0)\hat{\phi}_2(a)d^*bd^*a\\&=\int\phi_1(b)d^*b\int j_{\pi,\psi}(t(a)w_0)\hat{\phi}_2(a)d^*a.\end{aligned} \tag{8.5}$$

To show that the integral converges absolutely we notice that $j_{\pi,\psi}(t(a)w_0)$ is of order $|a|^{1/4}$ when $|a|$ is large and at most of order $|a|^{1/4}$ when $|a|$ is small. We can find positive constants A, B such that $\phi_1 \leq c\Upsilon_{A,B}$, where c is a positive constant and $\Upsilon_{A,B}$ is a characteristic function of the set $\{x : A \leq |x| \leq B\}$. Hence

$$\int |\phi_1(ba)j_{\pi,\psi}(t(a)w_0)\hat{\phi}_2(a)|d^*a \leq c' \int_{|A/b|}^{|B/b|} |a|^{-3/4}|\hat{\phi}_2(a)|da.$$

Since $\hat{\phi}_2(a)$ is rapidly decreasing when $|a|$ is large, it follows that the value of the integral is rapidly decreasing when $|b|$ is small. When $|b|$ is large the integral is of the order of $|b|^{-1/4}$. It is now clear that integrating further with $d^*b = |b|^{-1}db$ will converge absolutely both at 0 and ∞. ∎

LEMMA 8.2: *Let $f\colon U_2 \to \mathbf{C}$ be given by (8.3). Then*

$$\int_G f(g)i_{\pi,\psi}(g)dg = \int \phi_1(b)d^*b \int \phi_3(c)d^*c \int \phi_4(y)\psi(y)dy \int i_{\pi,\psi}(n(x)w_0)\phi_2(x)dx.$$

Proof: Integrating as in the proof of Lemma 8.1 and using the invariance properties of $i_{\pi,\psi}$, the proof is obvious. ∎

PROPOSITION 8.3: *$i_{\pi,\psi}(n(x)w_0)$ is the λ-Fourier transform of the function $\frac{1}{L(\pi,1/2)}|a|^{-1}j_{\pi,\psi}(t(a)w_0)$.*

Proof: It follows from (8.2), Lemma 8.1 and Lemma 8.2 that

$$\frac{1}{L(\pi,1/2)} \int j_{\pi,\psi}(t(a)w_0)\hat{\phi}_2(a)d^*a = \int i_{\pi,\psi}(n(x)w_0)\phi_2(x)dx$$

for all $\phi_2 \in C_c^\infty(\mathbf{R}^*)$. After writing $d^*a = |a|^{-1}da$ the proposition follows from standard Fourier analysis. ∎

Using Proposition 8.3, Theorem 6.4 and formulas (25), (28) on p. 54 and (27), (30) on p. 110 of [6], we can now compute $i_{\pi,\psi}$ for every irreducible unitary representation $\pi = \pi(\mu, \mu^{-1})$ of G. For the detailed computations see Appendix 3. We give here the final results:

COROLLARY 8.4: *Let $\pi = \pi(\mu, \mu^{-1})$ and $\psi = \psi_\lambda$.*

(i) *If $\mu(x) = |x|^{d-1/2}\operatorname{sgn}(x)$ with $d \in \mathbf{N}$, then π is a discrete series representation and we have*

$$i_{\pi,\psi}(n(y)w_0) = \frac{1}{L(\pi,1/2)}(\operatorname{sgn}(y)i)^d \frac{\sqrt{2}|\lambda|\pi}{|y|^{1/2}} e^{\lambda\pi i/y} J_{d-1/2}(\pi|\lambda|/|y|).$$

(ii) *If* $\mu(x) = |x|^{ir}(\mathrm{sgn}(x)^{\eta}$ *with* $r \in \mathbf{R}$ *and* $\eta \in \{0,1\}$, *then*

$$i_{\pi,\psi}(n(y)w_0) = \frac{1}{L(\pi,1/2)} \frac{|\lambda|\pi}{\sin(\pi i r)\sqrt{2|y|}} e^{\pi\lambda i/y} e^{\mathrm{sgn}(\lambda y)(\pi i/4)} \times [((-1)^{\eta+1} e^{\mathrm{sgn}(\lambda y)(-\pi r/2)} + \mathrm{sgn}(\lambda y) i e^{\mathrm{sgn}(\lambda y)(\pi r/2)}) J_{ir}(\pi|\lambda|/|y|) + ((-1)^{\eta} e^{\mathrm{sgn}(\lambda y)(\pi r/2)} - \mathrm{sgn}(\lambda y) i e^{\mathrm{sgn}(\lambda y)(-\pi r/2)}) J_{-ir}(\pi|\lambda|/|y|)].$$

(iii) *If* $\mu(x) = |x|^{r}(\mathrm{sgn}(x)^{\eta}$ *with* $0 < r < 1/2$ *and* $\eta \in \{0,1\}$, *then*

$$i_{\pi,\psi}(n(y)w_0) = \frac{1}{L(\pi,1/2)} \frac{|\lambda|\pi}{\sin(\pi r)\sqrt{2|y|}} e^{\pi\lambda i/y} e^{\mathrm{sgn}(\lambda y)(\pi i/4)} \times [((-1)^{\eta+1} e^{\mathrm{sgn}(\lambda y)(r\pi i/2)} + \mathrm{sgn}(\lambda y) i e^{\mathrm{sgn}(\lambda y)(-\pi r i/2)}) J_{r}(\pi|\lambda|/|y|) + ((-1)^{\eta} e^{\mathrm{sgn}(\lambda y)(-\pi r i/2)} - \mathrm{sgn}(\lambda y) i e^{\mathrm{sgn}(\lambda y)(\pi r i/2)}) J_{-r}(\pi|\lambda|/|y|)].$$

9. $GL(2)$ double cover, notation and preliminaries

In the next part of this paper we will prove a regularity theorem for Bessel distributions on the double cover of $GL(2)$ and $SL(2)$. The method of proof is similar to the proof of the regularity theorem for Bessel distributions on $GL(2)$ which was employed in Section 7. In that proof we used a kernel formula and an inner product formula in the Kirillov model. Since such formulas are not known for the double cover case, we will need to obtain them before we can prove the regularity theorem. We start in this section with some notation and preliminaries on double covers.

For $a, b \in \mathbf{R}^*$ we denote by (a,b) the Hilbert symbol, which is defined by

$$(a,b) = \begin{cases} -1 & \text{if } a<0 \text{ and } b<0, \\ 1 & \text{otherwise.} \end{cases}$$

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and set

$$\chi(g) = \begin{cases} c & \text{if } c \neq 0; \\ d & \text{if } c = 0. \end{cases}$$

For $g_1, g_2 \in G = GL_2(\mathbf{R})$ we define the cocycle $\alpha(g_1,g_2)$ to be

$$\alpha(g_1,g_2) = \Big(\det(g_1), \frac{\chi(g_1g_2)}{\chi(g_1)}\Big)\Big(\frac{\chi(g_1g_2)}{\chi(g_1)}, \frac{\chi(g_1g_2)}{\chi(g_2)}\Big);$$

see ([12], p. 41). We let $\bar{G}$ be the metaplectic cover of G, i.e.,

$$\bar{G} = \{[g,\epsilon] : g \in G, \epsilon = \pm 1\}$$

with multiplication given by

$$[g_1,\epsilon_1][g_2,\epsilon_2] = [g_1g_2, \alpha(g_1,g_2)\epsilon_1\epsilon_2)].$$

The group $\bar{G}$ fits into the exact sequence

$$1 \to \{\pm 1\} \to \bar{G} \to G \to 1. \tag{9.1}$$

In general, if H is any subgroup of G we let $\bar{H}$ denote its full inverse image in G. In particular, if (9.1) splits over H, then $\bar{H}$ is the direct product of $\{\pm 1\}$ with H. We note in particular that (9.1) splits over N.

We let $S = SL_2(\mathbf{R})$ and $\bar{S}$ be the metaplectic cover of S viewed as a subgroup of $\bar{G}$. We shall identify an element $g \in G$ and $\epsilon = \pm 1$ with

$$g = [g,1] \in \bar{G}, \quad \epsilon = [e,\epsilon] \in \bar{G}. \tag{9.2}$$

We let N, A and B be subgroups of G as defined in Section 2. Let Z be the center of G. The group $\bar{Z}$ is Abelian but not central in $\bar{G}$. Let $z(c) = \operatorname{diag}(c,c)$. The commutation is given by

$$gz(c) = z(c)g(\det(g),c) \tag{9.3}$$

where $z(c)$ is identified with $[z(c),1]$ in $\bar{Z}$, $g = [g,1]$ is in $\bar{G}$ and (x,y) is the Hilbert symbol of $x,y \in \mathbf{R}^*$.

Let $\bar{K}$ be the group of elements of the form

$$\left[\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}, \epsilon\right].$$

We can identify the group $\mathbf{R}/4\pi\mathbf{Z}$ with $\bar{K}$ via a map $r\colon \mathbf{R} \to \bar{K}$ such that

$$r(\pi) = \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1\right]. \tag{9.4}$$

10. Genuine representations of $\overline{SL_2(\mathbf{R})}$

We recall some results of [7] on the classification of genuine irreducible unitary representations of $\bar{S}$.

Let σ be an irreducible unitary representation of $\bar{S}$ on a Fréchet space $\mathcal{F}$. We say that σ is genuine if

$$\sigma(\epsilon)v = \epsilon v, \quad \text{for } \epsilon = \pm 1 \text{ and all } v \in \mathcal{F}.$$

As in [21] we will consider the family $\sigma_{\eta,s}$ of induced representations:

Definition 10.1: Let $\eta \in \{0,1\}$ and $s \in \mathbf{C}$. Let $\sigma_{\eta,s}$ be the representation of $\bar{S}$ on the space of smooth functions $f\colon \bar{S} \to \mathbf{C}$ satisfying

$$f(\epsilon n(x)s(a)h) = \epsilon\gamma(a)(\mathrm{sgn}(a))^{\eta}|a|^{s+1}f(h) \tag{10.1}$$

where $x \in \mathbf{R}$, $a \in \mathbf{R}^*$, $\epsilon = \pm 1$, $h \in \bar{S}$ and

$$\gamma(a) = \begin{cases} 1 & \text{if } a > 0; \\ i & \text{if } a < 0. \end{cases}$$

We denote by $\mathcal{F}_{\sigma_{\eta,s}}$ the space of $\sigma_{\eta,s}$, that is, the space of functions satisfying (10.1).

Notice that $\bar{S}$ acts on $\mathcal{F}_{\sigma_{\eta,s}}$ by right translations via the formula

$$(\sigma_{\eta,s}(h_1)f)(h) = f(hh_1).$$

The Lie algebra $\mathrm{Lie}(\bar{S}) = \mathfrak{s} = \mathfrak{sl}_2$ acts on $\sigma_{\eta,s}$ by left invariant differential operators via the formula

$$(Df)(h) = \frac{d}{dt}(f(he^{tD}))|_{t=0}, \quad D \in \mathfrak{s}. \tag{10.2}$$

We let $\mathcal{V}(\eta,s)$ be the space of functions V_f, $f \in \sigma_{\eta,s}$ defined by

$$V_f(x) = f(wn(x)).$$

Since the mapping $f \mapsto V_f$ is injective, the space $\mathcal{V}(\eta,s)$ admits an $\bar{S}$ and $\mathfrak{s}$ action which is isomorphic to $\sigma_{\eta,s}$. In particular, we will need the following actions on $V \in \mathcal{V}(\eta,s)$:

$$(wV)(x) = \gamma(x)(\mathrm{sign}(x))^{\eta+1}|x|^{-s-1}V(-x^{-1}), \tag{10.3}$$

$$(XV)(x) = \frac{d}{dx}(V(x)), \tag{10.4}$$

where X is defined in Section 2. It follows from (10.3) that

$$V(x) = O(|x|^{-s-1}) \tag{10.5}$$

for $|x|$ large and $V \in \mathcal{V}(\eta,s)$. By (10.4) the derivative of V is also in $\mathcal{V}(\eta,s)$, hence inductively we have

$$\frac{d^n}{dx^n}(V(x)) = O(|x|^{-s-1}) \tag{10.6}$$

for $|x|$ large and $V \in \mathcal{V}(\eta,s)$. We define the Fourier transform of V by

$$\hat{V}(y) = \hat{V}^1(y) = \int V(x)e^{-2\pi iyx}dx, \tag{10.7}$$

where dx is now the standard Lebesgue measure on $\mathbf{R}$. By (10.5) this integral converges absolutely when $\mathrm{Re}(s) > 0$. Since $V(x)$ and all its derivatives are in $L^1(\mathbf{R})$, it follows from standard Fourier analysis that:

LEMMA 10.2: *Let $s \in \mathbf{C}$ and assume that $\mathrm{Re}(s) > 0$. Let $V \in \mathcal{V}(\eta, s)$. Then $\hat{V}(y)$ is a continuous, bounded and rapidly decreasing function on $\mathbf{R}$. Moreover,*

$$V(x) = \int \hat{V}(y) e^{2\pi i y x} dy. \tag{10.8}$$

Finally, we shall consider another model for $\sigma_{\eta,s}$, which we denote by $\hat{\mathcal{V}}(\eta, s)$ consisting of functions $\hat{V}$, $V \in \mathcal{V}(\eta, s)$. When $\mathrm{Re}(s_0) = 0$ and $\phi \in \mathcal{V}(\eta, s_0)$ we define $\hat{\phi}$ using analytic continuation as follows: We choose a section $V_s \in \mathcal{V}(\eta, s)$ as in [16] such that $V_{s_0} = \phi$. This is done by fixing η and taking a section $f_s \in V_{\sigma_{\eta,s}}$ such that f_s restricted to $\bar{K}$ is a smooth function independent of s. In other words, a section f_s is such that there exists a smooth function p on $\mathbf{R}/4\pi\mathbf{Z}$ such that $f_s(r(\theta)) = p(\theta)$ (see (9.4) for the definition of $r(\theta)$).

At this point, $\hat{V}_s$ is defined for $\mathrm{Re}(s) > 0$ by (10.7). By [16], $\hat{V}_s(y)$ is analytic in s and has analytic continuation to $\mathbf{C}$. (We will give another proof below.)

It is clear from (10.5) and from (10.6) that if $\mathrm{Re}(s) > 0$, then $\hat{V}$ is bounded and rapidly decreasing at ∞. We would like to get some uniform bounds in s for the Fourier transform of a section V_s:

THEOREM 10.3: *Let $f_s \in \mathcal{F}_{\sigma_{\eta,s}}$ be a section and let $V_s = V_{f_s}$. Let s be in a compact set Q in the region $0 \le \mathrm{Re}(s) \le 1/4$. Then there exists a positive constant C (independent of $s \in Q$ and y) such that*

$$|\hat{V}_s(y)| < C|y|^{-1/8}, \quad s \in Q.$$

Also, for every integer $n > 0$ there exists a constant $C_n > 0$ independent of $s \in Q$ such that

$$|\hat{V}_s(y)| < C_n |y|^{-n}, \quad y \in \mathbf{R},\ s \in Q.$$

Proof: Since f_s is a section, we can write

$$f_s(s(a)n(y)r(\theta)) = \gamma(a)(\mathrm{sgn}(a))^{\eta}|a|^{s+1}p(\theta)$$

where p is a smooth function as above. By [21] p. 23 we have

$$wn(x) = s((1+x^2)^{-1/2}))n(-x)r(\theta_x),$$

where $\theta_x \in (0, \pi)$ is determined by the equalities

$$\sin(\theta_x) = (1+x^2)^{-1/2}, \quad \cos(\theta_x) = -x(1+x^2)^{-1/2}.$$

It follows that

$$V_s(x) = (1+x^2)^{-s/2-1/2}p(\theta_x)$$

and that $V_s(x) << (1+x^2)^{-\operatorname{Re}(s)/2-1/2}$. It is easy to see that the derivatives of $V_s(x)$ in the x variable satisfy the same bound. Using a Taylor expansion for p, we can write $V_s(x) = p(\pi)(1+x^2)^{-s/2-1/2} + M_s(x)$ where $M_s(x)$ is a smooth function of x and the nth derivative of $M_s(x)$ satisfies

$$M_s^{(n)}(x) <<_n x^{-1}(1+x^2)^{-\operatorname{Re}(s)/2-1/2}, n \geq 0.$$

It follows that the Fourier transform of M_s is bounded uniformly in s when s is in a compact set in the region $\operatorname{Re}(s) > -1/2$ and that it is rapidly decreasing uniformly in s. Hence, the Fourier transform of $M_s(x)$ satisfies the requirements of the Lemma. It now remains to bound the Fourier transform of $P_s(x) = (1+x^2)^{-s/2-1/2}$. By ([6], 1.3 (7)) we have that

$$\hat{P}_s(y) = 2\sqrt{\pi}(\pi y)^{s/2}\Gamma(s/2+1/2)K_{s/2}(2\pi|y|).$$

Since $0 \leq \operatorname{Re}(s) \leq 1/4$ we have that $0 \leq \operatorname{Re}(s/2) \leq 1/8$. Hence, by Lemma 20.2 we have that for $s \in Q$,

$$|\hat{P}_s(y)| <<_Q |y|^{\operatorname{Re}(s/2)}|y|^{-1/8}e^{-2\pi|y|}.$$

It follows that if y is small then $|\hat{P}_s(y)| <<_Q |y|^{-1/8}$, and that when y is large, $|\hat{P}_s(y)|$ is rapidly decreasing uniformly in $s \in Q$. ∎

It is easy to describe the action of $s(a)$ and $n(x)$ on $\hat{\mathcal{V}}(\eta, s)$. One of our main goals is to describe the action of w on $\hat{\mathcal{V}}(\eta, s)$ which we will do in the next section. It is possible to do that for all the representations considered above or their irreducible subspaces. However, we are only interested in the unitary representations of $\bar{S}$. We now describe the genuine unitary dual of $\bar{S}$ following [7] and [21].

10.1. Principal Series. For $s = ir$, $r \in \mathbf{R}$, the representation $\sigma_{\eta,ir}$ is irreducible and unitary. These representations are called the principal series.

10.2. Complementary Series. For $s \in \mathbf{R}$, $0 < s < 1/2$, the representation $\sigma_{\eta,s}$ is irreducible and unitary. These representations are called the complementary series.

10.3. Discrete Series. Let $s \in -1/2 + \mathbf{N}$ and $\nu = \frac{1}{2}(-1)^\eta$. We distinguish two cases:

CASE 1: $s \in -\nu + 2\mathbf{Z}$. Then $\sigma_{\eta,s}$ has an irreducible unitary subspace. These representations are called the holomorphic discrete series.

CASE 2: $s \in \nu + 2\mathbf{Z}$. Then $\sigma_{\eta,s}$ has an irreducible unitary subspace. These representations are called the anti-holomorphic discrete series.

10.4. WEIL REPRESENTATIONS. For $s = -\frac{1}{2}$, the representation $\sigma_{\eta,s}$ has an irreducible subspace which is an r_ψ^+ Weil representation (see [22] p. 223). We ignore these representations completely in this paper.

Notation: For s as in the principal series, the complementary series or the discrete series, we denote by $\tilde{\sigma}_{\eta,s}$, and $\tilde{\mathcal{V}}(\eta, s)$ the unique irreducible subrepresentation and subspace respectively of $\sigma_{\eta,s}$ and $\mathcal{V}(\eta, s)$.

THEOREM 10.4 ([7], [21]): *Every genuine irreducible unitary representation of $\bar{S}$ is isomorphic to a unique principal series or a complementary series or a discrete series or a Weil representation. That is, if σ is a genuine irreducible unitary representation of $\bar{S}$ which is not isomorphic to r_ψ^+ then there exist unique s and η as above such that $\sigma \cong \tilde{\sigma}_{\eta,s}$.*

11. A kernel formula for the action of w

In this section we describe the action of the Weyl element w in the model $\hat{\tilde{\mathcal{V}}}_{s,\eta}$ when s is a complementary series parameter, a principal series parameter or a discrete series parameter. As in Section 10 we shall fix for the moment $\lambda = 1$, $\hat{V} = \hat{V}^1$ (see (2.2)) and $dx = dx_1$ to be the standard Lebesgue measure on $\mathbf{R}$.

Our aim is to find an explicit formula of the following nature:

$$w\hat{V}(b) = (wV)^\wedge(b) = \int k_{s,\eta}(a,b)\hat{V}(a)da \tag{11.1}$$

where $V \in \mathcal{V}(\eta, s)$ (or $\tilde{\mathcal{V}}(s, \eta)$ in the discrete series case) and $\hat{V}$ is defined by (10.7) when $\mathrm{Re}(s) > 0$ and by analytic continuation for $\mathrm{Re}(s) = 0$. In the case of the discrete series we will only prove the result for vectors whose Whittaker function satisfies a certain growth condition. We will show that all K finite vectors satisfy this growth condition. We believe that the growth condition is satisfied by every smooth vector in a discrete series representation.

To establish the formula we let $V \in \mathcal{V}_{\eta,s}$. Then for $\mathrm{Re}(s) > 0$ we have

$$w\hat{V}(b) = (wV)^\wedge(b) = \int \gamma(x)(\mathrm{sgn}(x))^{\eta+1}|x|^{-s-1}V(-x^{-1})e^{-2\pi ibx}dx. \tag{11.2}$$

It follows from Lemma 10.2 that when $\mathrm{Re}(s) > 0$ and $V \in \tilde{\mathcal{V}}(\eta, s)$, we can write

$$V(c) = \int \hat{V}(a) e^{2\pi i a c} da.$$

Combining this with (11.2) we get:

LEMMA 11.1: *Let s be such that $\mathrm{Re}(s) > 0$. Let $V \in \tilde{\mathcal{V}}(\eta, s)$. Then*

$$\begin{aligned} w\hat{V}(b) &= (wV)^{\wedge}(b) \\ &= \int \gamma(x)(\mathrm{sgn}(x))^{\eta+1}|x|^{-s-1}e^{-2\pi i b x}\left(\int \hat{V}(a)e^{-2\pi i a x^{-1}}da\right)dx. \end{aligned} \tag{11.3}$$

We would like to change the order of integration in the above equation. To do that we introduce a convergence factor as follows. If $V \in L^1(\mathbf{R})$, it follows from the dominated convergence theorem that

$$\hat{V}(b) = \lim_{\delta \to 0^+} \int V(x) e^{-2\pi i b x} e^{-\delta(|x|+|x|^{-1})} dx. \tag{11.4}$$

Hence, under the assumptions of Lemma 11.1 we have

$$w\hat{V}(b) = \tag{11.5}$$
$$\lim_{\delta \to 0^+} \int \gamma(x)(\mathrm{sgn}(x))^{\eta+1} e^{-\delta(|x|+|x|^{-1})}|x|^{-s-1}e^{-2\pi i b x}\left(\int \hat{V}(a)e^{-2\pi i a x^{-1}}da\right)dx.$$

Since $\hat{V} \in L^1(\mathbf{R})$, the iterated integral is absolutely convergent and we can interchange the order of integration to get:

LEMMA 11.2: *Let s be such that $\mathrm{Re}(s) > 0$. Let $V \in \tilde{V}_{\eta,s}$. Then*

$$w\hat{V}(b) =$$
$$\lim_{\delta \to 0} \int \hat{V}(a)\left(\int \gamma(x)(\mathrm{sgn}(x))^{\eta+1}|x|^{-s-1}e^{2\pi i(-ax^{-1}-bx)-\delta(|x|+|x|^{-1})}dx\right)da.$$

We write this equation in the form

$$w\hat{V}(b) = \lim_{\delta \to 0^+} \int \hat{V}(a) k_{s,\eta,\delta}(a,b) da \tag{11.6}$$

where

$$k_{s,\eta,\delta}(a,b) = \int \gamma(x)(\mathrm{sgn}(x))^{\eta+1}|a|^{-s-1}e^{2\pi i(-ax^{-1}-bx)-\delta(|x|+|x|^{-1})}dx. \tag{11.7}$$

We shall now compute $k_{s,\eta,\delta}(a,b)$ and prove that we can interchange the limit and integration in (11.6) to get the formula that was promised in (11.1).

11.1. A BRANCH OF THE LOG. We remove the ray $(-\infty, 0]$ from $\mathbf{C}$. For $z \in \mathbf{C} \setminus (-\infty, 0]$ we write $z = re^{i\theta}$ with $-\pi < \theta < \pi$ and $r > 0$. We let

$$\mathrm{Log}(z) = \mathrm{Log}(r) + i\theta, z^s = e^{s\,\mathrm{Log}(z)}.$$

In particular,

$$z^{1/2} = r^{1/2} e^{i\theta/2}. \tag{11.8}$$

11.2. COMPUTATION OF $k_{s,\eta,\delta}(a, y)$. The following formula is an easy consequence of a formula in [24] p. 182:

LEMMA 11.3: *Let $a_1, a_2 \in \mathbf{C}$ be such that $\mathrm{Re}(a_1) > 0$ and $\mathrm{Re}(a_2) > 0$. Then*

$$\begin{aligned} K(s, a_1, a_2) &= \int_0^\infty x^{-s-1} e^{-a_1/x - a_2 x} dx \\ &= \left(\frac{a_1}{a_2}\right)^{-s/2-1/2} \left|\frac{a_1}{a_2}\right|^{1/2} 2K_s(2a_1^{1/2} a_2^{1/2}) \end{aligned} \tag{11.9}$$

where K_s is the classical K Bessel function and the square root, and power is taken as in (11.1).

LEMMA 11.4: *Assume that s is in a compact set Q in $\mathbf{C}$. Then*

$$|K(s, a_1, a_2)| <<_Q \left|\frac{a_1}{a_2}\right|^{-\mathrm{Re}(s)} |K_s(2a_1^{1/2} a_2^{1/2})|.$$

Proof: This follows from the fact that $|z^\nu| = |z|^{\mathrm{Re}(\nu)} e^{-\arg(z)\,\mathrm{Im}(\nu)}$. Since $|\arg(z)| < \pi$ and $\mathrm{Im}(\nu)$ is bounded, we get that $|z^\nu| << |z|^{\mathrm{Re}(\nu)}$. ∎

We separate the integral in (11.7) into two parts:

$$k_{s,\eta,\delta}(a, b) = k^+_{s,\eta,\delta}(a, b) + k^-_{s,\eta,\delta}(a, b)$$

where

$$\begin{aligned} k^+_{s,\eta,\delta}(a, b) &= \int_0^\infty \gamma(x)(\mathrm{sgn}(x))^{\eta+1} |x|^{-s-1} e^{2\pi i(-ax^{-1} - bx) - \delta(|x| + |x|^{-1})} dx \\ &= \int_0^\infty x^{-s-1} e^{(-2\pi i a - \delta)x^{-1} + (-2\pi i b - \delta)x} dx \\ &= K(s, \delta + 2\pi i a, \delta + 2\pi i b) \end{aligned}$$

and

$$\begin{aligned} k^{-}_{s,\eta,\delta}(a,b) &= \int_{-\infty}^{0} \gamma(x)(\operatorname{sgn}(x))^{\eta+1}|x|^{-s-1}e^{2\pi i(-ax^{-1}-bx)-\delta(|x|+|x|^{-1})}dx \\ &= \int_{-\infty}^{0} i(-1)^{\eta+1}|x|^{-s-1}e^{(-2\pi ia+\delta)x^{-1}+(-2\pi ib+\delta)x}dx \\ &= i(-1)^{\eta+1}\int_{0}^{\infty} x^{-s-1}e^{(2\pi ia-\delta)x^{-1}+(2\pi ib-\delta)x}dx \\ &= i(-1)^{\eta+1}K(s,\delta-2\pi ia,\delta-2\pi ib) \end{aligned}$$

LEMMA 11.5: *Fix s with $0 < \operatorname{Re}(s)$ and fix $b \neq 0$. Assume that $\delta < 1$. Then for $|a|$ small*

$$|k_{s,\eta,\delta}(a,b)| << |a|^{-\operatorname{Re}(s)}, \tag{11.10}$$

and for $|a|$ large

$$|k_{s,\eta,\delta}(a,b)| << |a|^{-1/4}, \tag{11.11}$$

where the implied constants are independent of $\delta < 1$.

Proof: By Lemma 20.2 we have that $|K_s(z)| << |z|^{-\operatorname{Re}(s)}$. Hence, using Lemma 11.4 we get that

$$|k_{s,\eta,\delta}(a,b)| << |\delta+2\pi ia|^{-\operatorname{Re}(s)} + |\delta-2\pi ia|^{-\operatorname{Re}(s)} << |2\pi ia|^{-\operatorname{Re}(s)}.$$

To get the second bound we notice that by Corollary 20.4 and Lemma 11.4,

$$|k_{s,\eta,\delta}(a,b)| << |\delta+2\pi ia|^{-\operatorname{Re}(s)-1/4}|\delta+2\pi ib|^{\operatorname{Re}(s)-1/4}$$

when $|a|$ is large. When $\operatorname{Re}(s) \leq 1/4$ we have that

$$|\delta+2\pi ib|^{\operatorname{Re}(s)-1/4} \leq |2\pi ib|^{\operatorname{Re}(s)-1/4} = O(1).$$

When $\operatorname{Re}(s) > 1/4$ and $\delta < 1$ then

$$|\delta+2\pi ib|^{\operatorname{Re}(s)-1/4} \leq |1+2\pi ib|^{\operatorname{Re}(s)-1/4} = O(1).$$

When $|a|$ is large and $\delta < 1$ then $|\delta+2\pi ia| = O(|a|)$ and we get the result. ∎

We will now apply dominated convergence in (11.6) to get the desired kernel formula.

11.3. Complementary series. We fix s with $0 < \mathrm{Re}(s) < 1$ and we fix $b \neq 0$. We assume that $0 < \delta < 1$. By (11.10), we have that $|k_{s,\eta,\delta}(a,b)| << |a|^{-\mathrm{Re}(s)}$. By Lemma 10.2, $\hat{V}(a) = O(1)$. It follows that we can use the dominated convergence theorem in (11.6) at 0. By (11.10), we have that $|k_{s,\eta,\delta}(a,b)| << |a|^{-1/4}$ when $|a|$ is large. By Lemma 10.2, $\hat{V}(a)$ is rapidly decreasing at ∞. It follows that we can use the dominated convergence theorem in (11.6) at ∞ to conclude that

$$w\hat{V}(b) = \int \hat{V}(a)(\lim_{\delta\to 0^+} k_{s,\eta,\delta}(a,b))da = \int \hat{V}(a)k_{s,\eta}(a,b)da \tag{11.12}$$

where

$$k_{s,\eta}(a,b) = \lim_{\delta\to 0^+} k_{s,\eta,\delta}(a,b).$$

Then for $0 < \mathrm{Re}(s) < 1$ we have

$$\begin{aligned} k_{s,\eta}(a,b) &= K(s, 2\pi ia, 2\pi ib) + i(-1)^{\eta+1}K(s, 2\pi ia, 2\pi ib) \\ &= \left|\frac{a}{b}\right|^{-s/2} \gamma_{a,b}^{-s-1} 2K_s(4\pi|ab|^{1/2}\tilde{\gamma}_{a,b}) + i(-1)^{\eta+1}\left|\frac{a}{b}\right|^{-s/2} \alpha_{a,b}^{-s-1} 2K_s(4\pi|ab|^{1/2}\tilde{\alpha}_{a,b}) \end{aligned} \tag{11.13}$$

where

$$\gamma_{a,b} = \begin{cases} i & \text{if } a > 0 \text{ and } b < 0; \\ -i & \text{if } a < 0 \text{ and } b > 0; \\ 1 & \text{otherwise;} \end{cases} \qquad \tilde{\gamma}_{a,b} = \begin{cases} i & \text{if } a > 0 \text{ and } b > 0; \\ -i & \text{if } a < 0 \text{ and } b < 0; \\ 1 & \text{otherwise;} \end{cases} \tag{11.14}$$

and

$$\alpha_{a,b} = \begin{cases} -i & \text{if } a > 0 \text{ and } b < 0; \\ i & \text{if } a < 0 \text{ and } b > 0; \\ 1 & \text{otherwise;} \end{cases} \qquad \tilde{\alpha}_{a,b} = \begin{cases} -i & \text{if } a > 0 \text{ and } b > 0; \\ i & \text{if } a < 0 \text{ and } b < 0; \\ 1 & \text{otherwise.} \end{cases} \tag{11.15}$$

Lemma 11.6: *Fix $b \neq 0$. Let s be in a compact set Q in the region $0 \leq \mathrm{Re}(s) \leq 1/4$. Then*

$$|k_{s,\eta}(a,b)| <<_Q |a|^{-1/4}. \tag{11.16}$$

Proof: It follows from the above formulas for $k_{s,\eta}(a,b)$ and from Lemma 20.2 that

$$|k_{s,\eta}(a,b)| <<_Q |a|^{-\mathrm{Re}(s)/2-1/8}.$$

Hence, when $|a|$ is small we get our result. From Lemma 20.1 we have that

$$|k_{s,\eta}(a,b)| <<_Q |a|^{-\mathrm{Re}(s)/2-1/2}.$$

Hence, we get our result for large $|a|$. ∎

11.4. PRINCIPAL SERIES. Assume that $0 < \mathrm{Re}(s) < 1$ and let $V_s \in \mathcal{V}_{\eta,s}$ be a section. By (11.12) we have that

$$w\hat{V}_s(b) = (wV)_s^{\wedge}(b) = \int k_{s,\eta}(a,b)\hat{V}_s(a)da$$

where $k_{s,\eta}(a,b)$ is given by (11.13). We would like to extend this formula to s_0 with $\mathrm{Re}(s_0) = 0$. To do that we note that for fixed b, the left hand side of the above equation is an analytic function in s. Hence we have

$$w\hat{V}_{s_0}(b) = \lim_{s\to s_0} \int k_{s,\eta}(a,b)\hat{V}_s(a)da \tag{11.17}$$

where the limit is taken on a compact path to s_0 coming from the right. We shall now use the dominated convergence to bring the limit inside the integral. By Theorem 10.3 we have that $|\hat{V}_s(a)| = O(|a|^{-1/8})$ when s is in a compact set in the region $0 \le \mathrm{Re}(s) \le 1/4$. By (11.16) we have for such s that $|k_{s,\eta}(a,b)| = O(|a|^{-1/4})$. Hence we can use the dominated convergence theorem at $a = 0$ in (11.17).

By Theorem 10.3 we have that $\hat{V}_s(a)$ is rapidly decreasing at ∞ uniformly for s in a compact region as above. By (11.16) we have for such s that $|k_{s,\eta}(a,b)| = O(|a|^{-1/4})$. Hence we can use the dominated convergence theorem at $a = \infty$ in (11.17).

We conclude that (11.1) holds for $s = ir$, $r \in \mathbf{R}$ with $k_{s,\eta}(a,b)$ given by (11.13).

11.5. DISCRETE SERIES. Our starting point is (11.12), which is valid for $\mathrm{Re}(s) > 0$ and, in particular, for $s \in -1/2 + \mathbf{N}$. We fix such s. We will further assume that $V \in \tilde{\mathcal{V}}(\eta, s)$ (see Notation 10.4). In order to apply the dominated convergence theorem we will need the following lemma:

LEMMA 11.7: *Let $s \in -1/2 + \mathbf{N}$ and $V \in \tilde{V}(\eta, s)$ be a K-finite vector. Then $|\hat{V}(a)| = O(|a|^s)$ when $|a| > 0$ is small.*

Proof: Assume first that $s \in -\nu + 2\mathbf{Z}$, that is, the holomorphic discrete series case. Then $\hat{V}(a) = 0$ if $a < 0$ and the Lemma is true for such a. Let $a > 0$. We have that

$$\hat{V}(a) = a^{(s-1)/2} W_V(s(\sqrt{a})) \tag{11.18}$$

where W_V is the Whittaker function associated to V. By [21] Lemme 12, it follows that if V is the lowest weight in the holomorphic discrete series then

$$W_V(s(a)) = O(a^{s+1}) \tag{11.19}$$

when $a > 0$ is small. By applying the differential operator X repeatedly to the lowest weight vector, it is easy to see that all K-finite vectors satisfy (11.19).

Now the lemma follows from (11.18).

Anti-holomorphic discrete series: If $s \in \nu + 2\mathbf{Z}$, then $\hat{V}(a) = 0$ if $a > 0$ and, for $a < 0$,

$$\hat{V}(a) = |a|^{(s-1)/2} W_V(s(\sqrt{|a|}))$$

for some Whittaker function W_V. Using the formulas in [21] Lemme 12 for the maximal weight and applying the same arguments as before, we again reach the same conclusion. ∎

We are now ready to apply the dominated convergence theorem. Let $s \in -\nu + 2\mathbf{Z}$ and V be as in the Lemma above. Since $\hat{V}(a) = 0$ for $a < 0$ we only need to worry about the positive integration. When a is large, $\hat{V}(a)$ is rapidly decreasing and $k_{s,\eta,\delta}(a,b)$ is bounded independent of δ. Let V be such that $\hat{V}(a) = O(|a|^s)$ when $|a|$ is small (this includes all the K-finite vectors by the above lemma). Since $k_{s,\eta,\delta}(a,b) = O(|a|^{-s})$ independent of $\delta < 1$, we can apply the dominated convergence theorem to (11.12) to get

$$w\hat{V}(b) = \int \hat{V}(a)(\lim_{\delta \to 0^+} k_{s,\eta,\delta}(a,b))da.$$

We again let $k_{s,\eta}(a,b) = \lim_{\delta \to 0^+} k_{s,\eta,\delta}(a,b)$ and get that $k_{s,\eta}(a,b)$ is given by (11.13). A similar argument for the anti-holomorphic discrete series will yield the same formulas.

We now compute $k_{s,\eta}(a,b)$ explicitly for the holomorphic discrete series and the anti-holomorphic discrete series.

11.6. Holomorphic Discrete Series. Assume that $s \in -\frac{1}{2} + \mathbf{N}$ and $s \in -\nu + 2\mathbf{Z}$. Here $\nu = \pm\frac{1}{2}$ and $\eta \in \{0, 1\}$ is determined by the equation $\nu = \frac{1}{2}(-1)^\eta$. It follows that $(-1)^\eta = (-1)^{s+1/2}$. If $V \in \tilde{\mathcal{V}}(\eta, s)$, then $\hat{V}(a) = 0$ for $a < 0$ and $w\hat{V}(b) = 0$ for $b < 0$. Hence we can choose $k_{s,\eta}(a,b) = 0$ for $a < 0$ or $b < 0$. Now assume $a > 0$ and $b > 0$. Then

$$\begin{aligned} k_{\eta,s}(a,b) =& 2(a/b)^{-s/2} K_s(4\pi i\sqrt{ab}) + (-1)^{s+3/2} i K_s(-4\pi i\sqrt{ab}) \\ =& 2(a/b)^{-s/2} \frac{(-\pi i/2)e^{-s\pi i/2}}{i\sin(\pi s)} (e^{s\pi i} J_s(4\pi\sqrt{ab}) - J_{-s}(4\pi\sqrt{ab})) \\ &+ 2(a/b)^{-s/2}(-1)^{s+3/2} i \frac{(\pi i/2)e^{s\pi i/2}}{i\sin(\pi s)} (-e^{-s\pi i} J_s(4\pi\sqrt{ab}) + J_{-s}(4\pi\sqrt{ab})). \end{aligned}$$

Since $s \in -\frac{1}{2} + \mathbf{N}$ we have that $e^{-s\pi i/2} = -e^{s\pi i/2}(-1)^{s+3/2} i$. Hence the coefficient of J_{-s} above vanishes and we have

$$(11.20)\quad \begin{aligned} k_{\eta,s}(a,b) &= (a/b)^{-s/2}\frac{\pi}{\sin\pi s}(-e^{s\pi i/2}+(-1)^{s+1/2}ie^{-s\pi i/2})J_s(4\pi\sqrt{ab}) \\ &= 2(a/b)^{-s/2}\frac{\pi}{\sin\pi s}e^{-3\pi is/2}J_s(4\pi\sqrt{ab}). \end{aligned}$$

11.7. Anti-holomorphic Discrete Series. Assume that $s \in -\frac{1}{2}+\mathbf{N}$ and $s \in \nu + 2\mathbf{N}$. Here $\nu = \pm\frac{1}{2}$ and $\eta \in \{0,1\}$ is determined by the equation $\nu = \frac{1}{2}(-1)^\eta$. It follows that $(-1)^\eta = (-1)^{s-1/2}$. If $V \in \tilde{V}(\eta,s)$ then $\hat{V}(a) = 0$ for $a > 0$ and $w\hat{V}(b) = 0$ for $b > 0$. Hence we can choose $k_{s,\eta}(a,b) = 0$ for $a > 0$ or $b > 0$. Now assume $a < 0$ and $b < 0$. Then

$$\begin{aligned} k_{s,\eta}(a,b) =&2(|a/b|)^{-s/2}(K_s(-4\pi i\sqrt{ab}) + (-1)^{s+1/2}iK_s(4\pi i\sqrt{ab}) \\ =&2(a/b)^{-s/2}\frac{(\pi i/2)e^{s\pi i/2}}{i\sin(\pi s)}(-e^{-s\pi i}J_s(4\pi\sqrt{ab}) + J_{-s}(4\pi\sqrt{ab})) \\ &+ 2|a/b|^{-s/2}(-1)^{s+1/2}i\frac{(-\pi i/2)e^{-s\pi i/2}}{i\sin(\pi s)}(e^{s\pi i}J_s(4\pi\sqrt{ab}) - J_{-s}(4\pi\sqrt{|ab|})). \end{aligned}$$

Since $s \in -\frac{1}{2}+\mathbf{N}$ we have that $e^{s\pi i/2} = -e^{-s\pi i/2}(-1)^{s+1/2}i$. Hence the coefficient of J_{-s} above vanishes and we have

$$(11.21)\quad \begin{aligned} k_{s,\eta}(a,b) &= |a/b|^{-s/2}\frac{\pi}{\sin\pi s}(-e^{-s\pi i/2}+(-1)^{s-1/2}ie^{s\pi i/2})J_s(4\pi\sqrt{|ab|}) \\ &= 2|a/b|^{-s/2}\frac{\pi}{\sin\pi s}e^{3\pi is/2}J_s(4\pi\sqrt{|ab|}). \end{aligned}$$

We now summarize our results in this section.

Theorem 11.8: *Let s be such that $0 \le \mathrm{Re}(s) < 1$ or such that $s \in -\frac{1}{2}+\mathbf{N}$. Let $\eta \in \{0,1\}$. Let $V \in V(\eta,s)$, and if $s \in -\frac{1}{2}+\mathbf{N}$ we further assume that $V \in \tilde{V}(\eta,s)$ and that $\hat{V}(a) = O(|a|^s)$. Then*

$$(11.22)\quad w\hat{V}(b) = (wV)^\wedge(b) = \int k_{s,\eta}(a,b)\hat{V}(a)da$$

where $k_{s,\eta}(a,b)$ is given by (11.13). Moreover, if $\tilde{\sigma}_{\eta,s}$ is a holomorphic discrete series, then we can choose $k_{s,\eta}(a,b) = 0$ for $a < 0$ or $b < 0$ and $k_{s,\eta}(a,b)$ is given by (11.20) for $a > 0$ and $b > 0$. If $\tilde{\sigma}_{\eta,s}$ is an anti-holomorphic discrete series, then we can choose $k_{s,\eta}(a,b) = 0$ for $a > 0$ or $b > 0$ and $k_{s,\eta}(a,b)$ is given by (11.21) for $a < 0$ and $b < 0$.

We end this section by a simple generalization of (11.22) which we will need later. Let $\lambda \in \mathbf{R}^*$. For $f \in L^1(\mathbf{R})$ we define the λ-Fourier transform by

$$\hat{f}^\lambda(y) = \int f(x)e^{-\pi i\lambda yx}d_\lambda x;$$

here, $d_\lambda x = |\lambda|^{1/2} d_1 x$ where $d_1 x$ is the standard Lebesgue measure on $\mathbf{R}$. Then

$$\hat{f}^1(y) = |\lambda|^{-1/2} \hat{f}^\lambda(y/\lambda). \tag{11.23}$$

Hence if V satisfies the conditions of Theorem 11.8, then we have

$$|\lambda|^{-1/2} w\hat{V}^\lambda(b/\lambda) = |\lambda|^{-1/2} \int k_{s,\eta}(a,b)\hat{V}^\lambda(a/\lambda) d_\lambda a.$$

Hence

$$w\hat{V}^\lambda(b) = \int |\lambda| k_{s,\eta}(\lambda a, \lambda b)\hat{V}^\lambda(a) d_\lambda a.$$

We define

$$k_{\lambda,\eta,s}(a,b) = |\lambda|^{1/2} k_{s,\eta}(\lambda a, \lambda b). \tag{11.24}$$

Hence, for V as in Theorem 11.8, we have

$$w\hat{V}^\lambda(b) = \int k_{\lambda,\eta,s}(a,b)\hat{V}^\lambda(a) da. \tag{11.25}$$

12. Genuine representations of $\overline{GL_2(\mathbf{R})}$

We recall some facts from [7]. Let $\bar{G} = \overline{GL_2(\mathbf{R})}$ and let $\bar{G}^* = \{[g,\epsilon] | \det(g) > 0\}$. Its center is $\bar{Z}$. Let σ be an irreducible admissible genuine representation of $\bar{S}$. Let $\mathcal{F}_\sigma$ be the space of σ. Let μ be a character of $\bar{Z}$ whose restriction to $\bar{Z} \cap \bar{S}$ agrees with the central character of σ. We can extend σ to a representation $\mu \times \sigma$ of $\bar{G}^* = \bar{Z}\bar{S}$ given by

$$\mu \times \sigma(\bar{z}\bar{s}) = \mu(\bar{z})\sigma(\bar{s}).$$

PROPOSITION 12.1 ([7]): *The representation $\pi = \mathrm{Ind}(\mu \times \sigma, \bar{G}^*, \bar{G})$ is irreducible. Moreover, every irreducible admissible genuine representation π of $\bar{G}$ is of the form $\pi = \mathrm{Ind}(\mu \times \sigma, \bar{G}^*, \bar{G})$ for some irreducible admissible genuine representation σ of $\bar{S}$ and some character μ of $\bar{Z}$.*

Let $\mathcal{F}_\pi$ be the space of π. It is easy to describe the unitary representations of $\bar{G}$ via the unitary representations of $\bar{S}$.

LEMMA 12.2: *Let σ be an irreducible admissible genuine representation of $\bar{S}$. Then $\pi = \mathrm{Ind}(\mu \times \sigma, \bar{G}^*, \bar{G})$ is unitary if and only if σ and μ are unitary. In that case, the G invariant hermitian form on $\mathcal{F}_\pi$ is given (up to a scalar) by*

$$< F_1, F_2 >_{\bar{G}} = < F_1(e), F_2(e) >_{\bar{S}} + < F_1(e_1), F_2(e_1) >_{\bar{S}}$$

where $<,>_{\bar{S}}$ is the invariant inner product on $\mathcal{F}_\sigma$ and e, e_1 are defined in Section 2. Here F_1, F_2 are in the induced space $\mathcal{F}_\pi$, that is, F_i: $\bar{G} \to \mathcal{F}_\sigma$ is smooth and satisfies $F_i(g^*\bar{g}) = \mu \times \sigma(g^*)F_i(\bar{g})$ where $g^* \in \bar{G}^*$, $\bar{g} \in \bar{G}$ and $i = 1, 2$.

Remark 12.3: If $\mathcal{F}_\sigma$ is given as a space of function $\mathcal{F}_\sigma = \{f_\alpha\}$ where f_α: $\bar{S} \to \mathbf{C}$ and the action of $\bar{S}$ is by right translations, then we can realize $\pi = \operatorname{Ind}(\mu \times \sigma, \bar{G}^*, \bar{G})$ as a space of smooth functions F: $\bar{G}^* \times \bar{G} \to \mathbf{C}$ satisfying:

(1) $\bar{h} \mapsto F(\bar{h}, \bar{g})$ is in $\mathcal{F}_\sigma$ for every fixed $\bar{g} \in \bar{G}$.

(2) $F(\bar{z}g^*, \bar{g}) = \mu(\bar{z})F(g^*, \bar{g})$ for all $\bar{z} \in \bar{Z}, g^* \in \bar{G}^*$ and $\bar{g} \in \bar{G}$.

(3) $F(g_1^* g_2^*, \bar{g}) = F(g_1^*, g_2^*\bar{g})$ for all $g_1^*, g_2^* \in \bar{G}^*$ and $\bar{g} \in \bar{G}$.

13. A kernel formula for the action of w in the Whittaker model

We use the results in Section 11 to derive a kernel formula for the Weyl element in the Whittaker model of a genuine irreducible unitary representation of $\bar{G}$. This formula is an analog of the kernel formula in [5] for unitary representations of G which we reproduced in (6.1). We remark that the same method can be applied to prove the formula of Cogdell and Piatetski-Shapiro. We carry out this process for G in Appendix 2.

Let $\tilde{\sigma}_{\eta,s}$ be an irreducible unitary representation of $\bar{S}$ as defined in (10.4). That is, assume that either $s = ir$, $r \in \mathbf{R}$ or $s \in \mathbf{R}$ and either $0 < s < 1/2$ or $s \in -1/2 + \mathbf{N}$. Moreover, if $s \in -1/2 + \mathbf{N}$ then we let $\mathcal{F}_{\tilde{\sigma}_{\eta,s}}$ be the space of functions defined in 10.4.

Let μ be a unitary character of $\bar{Z}$ extending the central character of $\tilde{\sigma}_{\eta,s}$. Let $\pi(\mu, \eta, s) = \operatorname{Ind}(\mu \times \tilde{\sigma}_{\eta,s}, \bar{G}^*, \bar{G})$. Then by Remark 12.3, $\mathcal{F}_{\pi(\mu,\eta,s)}$ is the space of functions F: $\bar{G}^* \times \bar{G} \to \mathbf{C}$ satisfying $\bar{h} \mapsto F(\bar{h}, \bar{g})$ is in $\mathcal{F}_{\tilde{\sigma}_{\eta,s}}$ for every fixed $\bar{g} \in \bar{G}$. In particular,

$$F(\bar{z}s(a)n(x)g^*, g) = \mu(\bar{z})\gamma(a)|a|^{s+1}(\operatorname{sgn}(a))^\eta F(g^*, g). \tag{13.1}$$

For $F \in \mathcal{F}_{\pi(\mu,\eta,s)}$ we define

$$V_{F,e}(x) = F(wn(x), e), \quad V_{F,e_1}(x) = F(wn(x), e_1). \tag{13.2}$$

13.1. Whittaker models for $\pi(\mu, \eta, s)$. Fix $\lambda \in \mathbf{R}^*$. Let

$$\psi_\lambda(x) = e^{2\pi i \lambda x}$$

and let ψ be a character of N defined by

$$\psi(n(x)) = \psi_\lambda(x).$$

Let π be an irreducible unitary genuine representation of $\tilde{G}$ on a Fréchet space $\mathcal{F} = \mathcal{F}_\pi$ and let μ be a character of $\bar{Z}$ extending the central character of π. Let L be a continuous linear functional on $\mathcal{F}$. We say that L is a (ψ, μ) Whittaker functional on $\mathcal{F}$ if

$$L(\pi(\bar{z})\pi(n)v) = \mu(z)\psi(n)L(v), \quad \text{for all } \bar{z} \in \tilde{Z},\ n \in N,\ v \in \mathcal{F}.$$

It is well known that the space of such functionals is at most one-dimensional. Given a (ψ, μ) Whittaker functional L on $\mathcal{F}$ we let

$$W_v(\bar{g}) = L(\pi(\bar{g})v), \quad g \in \bar{G},\ v \in \mathcal{F}.$$

Then $\bar{G}$ acts on the space of such functions by right translations. This is called the Whittaker model of π.

We now construct such functionals for $\pi(\mu, \eta, s)$.

For $F \in \mathcal{F}_{\pi(\mu,\eta,s)}$ we define

$$L^+(F) = \int F(wn(x), e)\psi_\lambda(-x)dx, \quad L^-(F) = \int F(wn(x), e_1)\psi_\lambda(x)dx.$$

Here $dx = d_\lambda x$. The integrals above are absolutely convergent if $\mathrm{Re}(s) > 0$ and they are defined by analytic continuation in s otherwise (see Section 10). It is easy to check that L^+ is a (ψ, μ^+) Whittaker functional and that L^- is a (ψ, μ^-) Whittaker functional where

$$\mu^+ = \mu, \mu^-(z(c)) = \mathrm{sign}(c)\mu(z(c)). \tag{13.3}$$

We remark that L^+ or L^-, but not both, could be identically zero on $\mathcal{F}_{\pi(\mu,\eta,s)}$. When these functionals are nonzero we obtain two Whittaker models for $\pi(\mu, \eta, s)$: One given by the functions

$$W_F^+(\bar{g}) = L^+(\pi(\bar{g})F)$$

and the other obtained by the functions

$$W_F^-(\bar{g}) = L^-(\pi(\bar{g})F).$$

In particular, we have

$$W_F^+(t(a)) = \int F(wn(x), t(a))\psi_\lambda(-x)dx, \tag{13.4}$$

$$W_F^-(t(a)) = \int F(wn(x), t(-a))\psi_\lambda(x)dx.$$

If $a > 0$ then we have

$$
\begin{aligned}
W_F^+(t(a)) &= \int F(wn(x)t(a), e)\psi_\lambda(-x)dx \\
&= \int F(wt(a)n(a^{-1}x), e)\psi_\lambda(-x)dx \\
&= \int F(z(\sqrt{a})s(\sqrt{a}^{-1})wn(x), e)\psi_\lambda(-ax)adx \\
&= \mu(\sqrt{a})a^{(1-s)/2}\hat{V}_{F,e}^{\lambda}(a).
\end{aligned} \tag{13.5}
$$

Here $\hat{V}_{F,e}^{\lambda}$ is the λ-Fourier transform of $V_{F,e}$ (see (2.3).) We let $\chi(|a|) = \mu(\sqrt{|a|})|a|^{(1-s)/2}$. Then similar computations as in (13.5) will give

$$
W_F^+(t(a)) = \begin{cases} \chi(|a|)\hat{V}_{F,e}^{\lambda}(a) & \text{if } a > 0; \\ \chi(|a|)\hat{V}_{F,e_1}^{-\lambda}(a) & \text{if } a < 0; \end{cases} \tag{13.6}
$$

and

$$
W_F^-(t(a)) = \begin{cases} \chi(|a|)\hat{V}_{F,e_1}^{-\lambda}(a) & \text{if } a > 0; \\ \chi(|a|)\hat{V}_{F,e}^{\lambda}(a) & \text{if } a < 0. \end{cases} \tag{13.7}
$$

Our goal is to prove the existence of functions $k^{+,+}(a,b)$, $k^{+,-}(a,b)$, $k^{-,+}(a,b)$, $k^{-,-}(a,b)$ such that

$$
(wW_F^+)(t(b)) = \int k^{+,+}(a,b)W_F^+(t(a))da + \int k^{+,-}(a,b)W_F^-(t(a))da, \tag{13.8}
$$

$$
(wW_F^-)(t(b)) = \int k^{-,+}(a,b)W_F^+(t(a))da + \int k^{-,-}(a,b)W_F^-(t(a))da. \tag{13.9}
$$

This will follow easily from (13.6), (13.7) and (11.25). By (11.25) we have

$$
w\hat{V}_{F,e}^{\lambda}(b) = \int k_{\lambda,\eta,s}(a,b)\hat{V}_{F,e}^{\lambda}(a)da, \quad w\hat{V}_{F,e_1}^{-\lambda}(b) = \int k_{-\lambda,\eta,s}(a,b)\hat{V}_{F,e_1}^{-\lambda}(a)da.
$$

Hence if we choose

$$
k^{+,+}(a,b) = \begin{cases} \frac{\chi(|b|)}{\chi(|a|)}k_{\lambda,\eta,s}(a,b) & \text{if } a > 0 \text{ and } b > 0; \\ \frac{\chi(|b|)}{\chi(|a|)}k_{-\lambda,\eta,s}(a,b) & \text{if } a < 0 \text{ and } b < 0; \\ 0 & \text{if } ab < 0; \end{cases} \tag{13.10}
$$

$$
k^{+,-}(a,b) = \begin{cases} \frac{\chi(|b|)}{\chi(|a|)}k_{-\lambda,\eta,s}(a,b) & \text{if } a > 0 \text{ and } b < 0; \\ \frac{\chi(|b|)}{\chi(|a|)}k_{\lambda,\eta,s}(a,b) & \text{if } a < 0 \text{ and } b > 0; \\ 0 & \text{if } ab > 0; \end{cases} \tag{13.11}
$$

and similarly for $k^{-,+}$ and $k^{-,-}$, then (13.8) and (13.9) will be true.

LEMMA 13.1:

$$\mu(a^{-1})k^{+,+}(1,ab) = |a|k^{+,+}(a,b), \tag{13.12}$$

and the same is true if we replace $k^{+,+}$ with $k^{+,-}$, $k^{-,+}$ and $k^{-,-}$.

Proof: The proof is by direct computation. We will assume $\lambda > 0$ and prove the equality for $k^{+,+}$. For the other cases the proof is similar. By (13.10), we have that if $ab < 0$ then $k^{+,+}(1,ab) = k^{+,+}(a,b) = 0$. Hence we can assume that $ab > 0$. In that case

$$\begin{aligned}\mu(a^{-1})k^{+,+}(1,ab) =&\mu(a^{-1})\frac{\chi(|ab|)}{\chi(1)}k_{\lambda,\eta,s}(1,ab)\\ =&2\lambda^{1/2}\mu(a^{-1})\mu(\sqrt{|ab|}|ab|^{(1-s)/2}|ab|^{s/2}\\ &\times(K_s(4\pi i\lambda|ab|^{1/2})+i(-1)^{\eta+1}K_s(-4\pi i\lambda|ab|^{1/2}))\\ =&2\lambda^{1/2}\frac{\mu(\sqrt{|b|})}{\mu(\sqrt{|a|})}\mu(\text{sign}(a))|ab|^{1/2}\\ &\times(K_s(4\pi i\lambda|ab|^{1/2})+i(-1)^{\eta+1}K_s(-4\pi i\lambda|ab|^{1/2})).\end{aligned} \tag{13.13}$$

On the other hand, we have

$$|a|k^{+,+}(a,b) = \\ 2\lambda^{1/2}\frac{\mu(\sqrt{|b|})}{\mu(\sqrt{|a|})}|ab|^{1/2}(K_s(4\pi i\lambda|ab|^{1/2})+i(-1)^{\eta+1}K_s(-4\pi i\lambda|ab|^{1/2}))$$

if $a > 0$ and $b > 0$ while

$$|a|k^{+,+}(a,b) = \\ 2\lambda^{1/2}\frac{\mu(\sqrt{|b|})}{\mu(\sqrt{|a|})}|ab|^{1/2}(K_s(-4\pi i\lambda|ab|^{1/2})+i(-1)^{\eta+1}K_s(4\pi i\lambda|ab|^{1/2}))$$

if $a < 0$ and $b < 0$. Since

$$\mu(\text{sign}(a)) = \begin{cases} 1 & \text{if } a > 0, \\ i(-1)^{\eta} & \text{if } a < 0, \end{cases}$$

we get our equality. ∎

We now transform $k^{+,+}$, $k^{+,-}$, $k^{-,+}$, $k^{-,-}$ into functions on an open set of $\bar{G}$. Fix $\lambda \in R^*$ and let $\psi = \psi_\lambda$. Let $\pi = \pi(\mu,\eta,s)$. Let $j^{+,+}_{\pi,\psi}$, $j^{-,+}_{\pi,\psi}$, $j^{+,+}_{\pi,\psi}$, $j^{-,+}_{\pi,\psi}$ be functions on $\bar{B}w\bar{B}$ defined as follows:

$$j^{+,+}_{\pi,\psi}(t(a)w) = k^{+,+}(1,a), \tag{13.14}$$
$$j^{+,+}_{\pi,\psi}(\bar{z}_1 n(x)\bar{g}n(y)\bar{z}_2) = \mu^+(\bar{z}_1)\mu^+(\bar{z}_2)\psi_\lambda(x)\psi_\lambda(y)j^{+,+}_{\pi,\psi}(\bar{g}).$$

Here $\bar{g} \in \bar{B}w\bar{B}$; $j^{+,-}_{\pi,\psi}$, $j^{-,+}_{\pi,\psi}$ and $j^{-,-}_{\pi,\psi}$ are defined in the same way. It is easy to check that $j^{+,+}_{\pi,\psi}$ and the others are well defined. Our main theorem of this section is the following.

THEOREM 13.2: *Let $F \in \mathcal{F}_\pi(\mu,\eta,s)$. If π is a discrete series, we assume further that $W_F^+(t(a))$ and $W_F^-(t(a))$ are of the order $|a|^{1/2+s/2}$ when $|a|$ is small. (Notice that this condition is satisfied by the K-finite vectors). Then*

$$W_F^+(\bar{g}) = \int j^{+,+}_{\pi,\psi}(\bar{g}t(a^{-1}))W_F^+(t(a))d^*a + \int j^{+,-}_{\pi,\psi}(\bar{g}t(a^{-1}))W_F^-(t(a))d^*a,$$

$$W_F^-(\bar{g}) = \int j^{-,+}_{\pi,\psi}(\bar{g}t(a^{-1}))W_F^+(t(a))d^*a + \int j^{-,-}_{\pi,\psi}(\bar{g}t(a^{-1}))W_F^-(t(a))d^*a.$$

Proof: Let $\bar{g} = t(b)w$. Then

$$\begin{aligned} j^{+,+}_{\pi,\psi}(\bar{g}t(a^{-1})) &= j^{+,+}_{\pi,\psi}(t(b)wt(a^{-1})) = (a^{-1}b, a^{-1})j^{+,+}_{\pi,\psi}(z(a^{-1})t(ab)w) \\ &= (a^{-1}b,a^{-1})\mu(a^{-1})k^{+,+}(1,ab) = \mu(a^{-1})k^{+,+}(1,ab). \end{aligned}$$

The last equality follows from the fact that $k^{+,+}(1,ab) = 0$ if $ab < 0$ and that $(a^{-1}b, a^{-1}) = 1$ if $ab > 0$. By (13.12) we get that $j^{+,+}_{\pi,\psi}(\bar{g}t(a^{-1})) = |a|k^{+,+}(a,b)$. After obtaining similar equalities for $j^{+,-}_{\pi,\psi}$, $j^{-,+}_{\pi,\psi}$ and $j^{-,-}_{\pi,\psi}$ we can use (11.22) to get the desired formula ∎

Remark 13.3: The reader might find the transition from the kernel formulas of Section 11 to the kernel formulas in the theorem above a little cumbersome. Our opinion is that the kernel formula in the Whittaker model above is the more simple and natural of the two. We mention that the same Whittaker kernel formula appears in the p-adic case ([18], [4]) where it is proved differently. The main difference is that here we use the classification of irreducible unitary representations, while in the p-adic case only the existence of a Whittaker model is used. The reader is encouraged to compare the two proofs.

We end this section with explicit formulas for $j^{+,+}_{\pi,\psi}$ which we shall need later. These formulas are easy to obtain using (13.14), (13.10), (11.25), (11.13) and [14].

(1) $\pi = \pi(\mu,\eta,ir)$, $r \in \mathbf{R}$.

$a > 0$:

$$\begin{aligned} &j^{+,+}_{\pi,\psi}(t(a)w) \\ &\quad= \frac{\chi(|a|)}{\chi(1)}k_{\lambda,\eta,s}(1,a) \\ &\quad= \mu(|a|^{1/2})|a|^{(1-s)/2}|\lambda|^{1/2}k_{\eta,s}(\lambda,\lambda a) \end{aligned}$$

$$= 2|\lambda|^{1/2}\mu(\sqrt{|a|})|a|^{1/2}(K_{ir}(4\pi i\lambda|a|^{1/2}) + i(-1)^{\eta+1}K_{ir}(-4\pi i\lambda|a|^{1/2}))$$

$$= \begin{cases} |\lambda|^{1/2}\mu(\sqrt{|a|})|a|^{1/2}\frac{-\pi}{\sin(\pi ir)}[(e^{-(\pi r)/2} + i(-1)^{\eta+1}e^{(\pi r)/2})J_{ir}(4\pi|\lambda||a|^{1/2}) \\ +(-e^{(\pi r)/2} + i(-1)^{\eta}e^{-(\pi r)/2})J_{-ir}(4\pi|\lambda||a|^{1/2})] \\ \text{if } \lambda > 0; \\ |\lambda|^{1/2}\mu(\sqrt{|a|})|a|^{1/2}\frac{-\pi}{\sin(\pi ir)}[(e^{(\pi r)/2} + i(-1)^{\eta+1}e^{-(\pi r)/2})J_{ir}(4\pi|\lambda||a|^{1/2}) \\ +(-e^{-(\pi r)/2} + i(-1)^{\eta}e^{(\pi r)/2})J_{-ir}(4\pi|\lambda||a|^{1/2})] \\ \text{if } \lambda < 0; \end{cases}$$

$a < 0$:

$$j^{+,+}_{\pi,\psi}(t(a)w) = 0.$$

(2) $\pi = \pi(\mu, \eta, s)$, $s \in \mathbf{R}$, $0 < s < 1/2$.

$a > 0$:

$$j^{+,+}_{\pi,\psi}(t(a)w)$$

$$= \begin{cases} |\lambda|^{1/2}\mu(\sqrt{|a|})|a|^{1/2}\frac{-\pi}{\sin(\pi s)}[(e^{(\pi is)/2} + i(-1)^{\eta+1}e^{-(\pi is)/2})J_{s}(4\pi|\lambda||a|^{1/2}) \\ +(-e^{-(\pi is)/2} + i(-1)^{\eta}e^{(\pi is)/2})J_{-s}(4\pi|\lambda||a|^{1/2})] \\ \text{if } \lambda > 0; \\ |\lambda|^{1/2}\mu(\sqrt{|a|})|a|^{1/2}\frac{-\pi}{\sin(\pi s)}[(e^{-(\pi is)/2} + i(-1)^{\eta+1}e^{(\pi is)/2})J_{s}(4\pi|\lambda||a|^{1/2}) \\ +(-e^{(\pi is)/2} + i(-1)^{\eta}e^{-(\pi is)/2})J_{s}(4\pi|\lambda||a|^{1/2})] \\ \text{if } \lambda < 0; \end{cases}$$

$a < 0$:

$$j^{+,+}_{\pi,\psi}(t(a)w) = 0.$$

(3) $\pi = \pi(\mu, \eta, s)$, $s \in -1/2 + \mathbf{N}$, $s \in -\nu + 2\mathbf{N}$.

$$j^{+,+}_{\pi,\psi}(t(a)w)$$

$$= \begin{cases} 2|\lambda|^{1/2}\mu(\sqrt{|a|})|a|^{1/2}\frac{\pi}{\sin \pi s}e^{-3\pi is/2}J_s(4\pi|\lambda|\sqrt{|a|}) & \text{if } a > 0 \text{ and } \lambda > 0; \\ 0 & \text{otherwise.} \end{cases}$$

$$j^{-,-}_{\pi,\lambda}(t(a)w)$$

$$= \begin{cases} 2|\lambda|^{1/2}\mu(\sqrt{|a|})|a|^{1/2}\frac{\pi}{\sin \pi s}e^{-3\pi is/2}J_s(4\pi|\lambda|\sqrt{|a|}) & \text{if } a > 0 \text{ and } \lambda < 0; \\ 0 & \text{otherwise.} \end{cases}$$

(4) $\pi = \pi(\mu, \eta, s)$, $s \in -1/2 + \mathbf{N}$, $s \in \nu + 2\mathbf{N}$.

$$j^{+,+}_{\pi,\lambda}(t(a)w)$$

$$= \begin{cases} -2|\lambda|^{1/2}\mu(\sqrt{|a|})|a|^{1/2}\frac{\pi}{\sin \pi s}e^{3\pi is/2}J_s(4\pi|\lambda|\sqrt{|a|}) & \text{if } a > 0 \text{ and } \lambda < 0; \\ 0 & \text{otherwise.} \end{cases}$$

$$j^{-,-}_{\pi,\lambda}(t(a)w)$$

$$= \begin{cases} -2|\lambda|^{1/2}\mu(\sqrt{|a|})|a|^{1/2}\frac{\pi}{\sin \pi s}e^{-3\pi is/2}J_s(4\pi|\lambda|\sqrt{|a|}) & \text{if } a > 0 \text{ and } \lambda > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 13.4: In the holomorphic discrete series (case (3)) we have that when $\lambda > 0$, $j_{\pi,\lambda}^{+,-} = j_{\pi,\psi}^{-,+} = j_{\pi,\psi}^{-,-} = 0$, and when $\lambda < 0$, $j_{\pi,\psi}^{+,+} = j_{\pi,\psi}^{+,-} = j_{\pi,\psi}^{-,+} = 0$.

A similar situation occurs in the anti-holomorphic discrete series.

COROLLARY 13.5: *Let π be a discrete series representation. Then*

$$\overline{j_{\pi,\psi}^{+,+}(g)} = j_{\pi,\psi}^{+,+}(g^{-1}), \quad \overline{j_{\pi,\psi}^{-,-}(g)} = j_{\pi,\psi}^{-,-}(g^{-1}).$$

Proof: Using the invariance properties of $j_{\pi,\psi}^{+,+}$ it is enough to prove the equality for $g = t(a)w$. In this case $g^{-1} = w^{-1}t(a^{-1}) = z(-a^{-1})t(a)w$. Hence it is enough to prove that

$$\overline{j_{\pi,\psi}^{+,+}(t(a)w)} = \mu^+(-a)j_{\pi,\psi}^{+,+}(t(a)w)$$

for all $a \in \mathbf{R}^*$. This follows easily from the formulas above. ∎

COROLLARY 13.6: *Let π be a genuine irreducible unitary representation of $\bar{G}$. Then*

$$j_{\pi,\psi}^{\pm,\pm}(s(a)w_0) = O(|a|^{1/2}) \quad \text{and} \quad j_{\pi,\psi}^{\pm,\pm}(s(a)w) = O(|a|^{1/2})$$

when $|a|$ is large.

14. Local integrability of $j_{\pi,\psi}^{+,+}$ and $j_{\pi,\psi}^{-,-}$

Let π be a genuine irreducible unitary representation of $\bar{G}$ and fix a character $\psi = \psi_\lambda$ of N. Let $j_\pi^{+,+} = j_{\pi,\psi}^{+,+}$ and $j_\pi^{-,-} = j_{\pi,\psi}^{-,-}$ be the functions on $\bar{B}w\bar{B}$ defined in Section 13. We extend them to $\bar{G}$ by setting them to be identically zero outside of $\bar{B}w\bar{B}$.

THEOREM 14.1: *$j_\pi^{+,+}$ and $j_\pi^{-,-}$ are locally integrable on $\bar{G}$.*

Proof: We note that $\bar{G}$ is a union of two connected components:

$$U_1 = \{\bar{g} = [g,\epsilon] : \det(g) > 0\} \quad \text{and} \quad U_2 = \{\bar{g} = [g,\epsilon] : \det(g) < 0\}.$$

It is enough to prove that $j_\pi^{+,+}$ and $j_\pi^{-,-}$ are locally integrable on each connected component. We will prove that $j_\pi^{+,+}$ is locally integrable on U_1, the other cases being similar. Let $f \in C_c^\infty(\bar{G})$ be positive. We need to show that

$$\int_{U_1} |j_\pi^{+,+}(\bar{g})| f(\bar{g}) d\bar{g} < \infty$$

for every such f. Let $U' = U_1 \cap \bar{B}w\bar{B}$. Then U' is the set of elements of the form $\bar{g} = [g, \epsilon]$ with $g = n(x)s(a)z(c)wn(y)$. We write

$$\int_{U_1} |j_\pi^{+,+}(\bar{g})| f(\bar{g}) d\bar{g} = \int_{c>0, a\in\mathbf{R}^*, x,y\in\mathbf{R}} |j_\pi^{+,+}([n(x)s(a)z(c)wn(y), 1])| \times f([n(x)s(a)z(c)wn(y), 1])|a|^{-2} d^*a dx d^*c dy$$
$$+ \int_{c>0, a\in\mathbf{R}^*, x,y\in\mathbf{R}} |j_\pi^{+,+}([n(x)s(a)z(c)wn(y), -1]) \times |f([n(x)s(a)z(c)wn(y), -1])|a|^{-2} d^*a dx d^*c dy.$$

We shall show that the first summand is finite. Similar arguments show that the second is finite.

$$\int_{c>0, a\in\mathbf{R}^*, x,y\in\mathbf{R}} |j_\pi^{+,+}(n(x)s(a)z(c)wn(y))| f(n(x)s(a)z(c)wn(y))|a|^{-2} d^*a dx d^*c dy$$
$$= \int |j_\pi^{+,+}(s(a)w)| \Big(\int_{c>0, x,y\in\mathbf{R}} f(n(x)s(a)z(c)wn(y)) dx d^*c dy \Big) |a|^{-2} d^*a.$$

Let $O_{||}(f,a) = \int_{c>0,x,y\in\mathbf{R}} f(n(x)s(a)z(c)wn(y))dxd^*cdy$. By Proposition 4.1 we have that $O_{||}(f,a) = 0$ if $|a|$ is small and $O_{||}(f,a) = O(|a|)$ if $|a|$ is large. Using Corollary 13.6 we get that the above integral is finite. ∎

15. Inner product in the Whittaker model

Let $(\pi, \mathcal{F})$ be an irreducible genuine unitary representation of $\bar{G}$ with central character μ. Let μ^+ and μ^- be the two characters on $\bar{Z}$ extending μ. Let $\psi = \psi_\lambda$ be a nontrivial character of N. Let L^+ and L^- be (ψ, μ^+) and (ψ, μ^-) Whittaker functionals on $\mathcal{F}$ respectively and we assume that they are nonzero if possible.

Our aim in this section is to prove that there exist nonzero scalars c^+ and c^- such that

(15.1)
$$< F_1, F_2 >= c^+ \int W_{F_1}^+(t(a))\overline{W_{F_2}^+(t(a))} d^*a + c^- \int W_{F_1}^-(t(a))\overline{W_{F_2}^-(t(a))} d^*a$$

is a $\bar{G}$ invariant continuous inner product on $\mathcal{F}$. Here $F_1, F_2 \in \mathcal{F}$ and $W_F^\pm(g) = L^\pm(\pi(g)F)$ for $g \in \bar{G}$, $F \in \mathcal{F}$. In the discrete series case we will prove this formula for K-finite vectors and for vectors satisfying a certain growth condition. We believe that the formula is true for all smooth vectors.

The measure d^*a appearing above is $d^*_\lambda a$. For the moment we will fix $\lambda = 1$, $da = d_1 a$ the standard Lebesgue measure on R and $d^*a = da/|a|$.

Remark 15.1: The convergence of the above integrals follows from the fact that $W_F^{\pm}(t(a))$ is rapidly decreasing at ∞ and of the order of $|a|^\beta$ with $\beta > 0$ at zero. (See (13.6), (13.7), Theorem 10.3, Lemma 10.2 and Lemma 11.7.) In the case of the discrete series we shall assume that $W_{F_i}^{\pm}(t(a))$ is of the order of $|a|^\beta$ with $\beta > 0$ at zero. It is also clear that $< F_1, F_2 >$ is continuous and invariant under $\bar{B}$, the upper triangular Borel of $\bar{G}$, for every c^+ and c^- and all such vectors. Thus it remains to prove the existence of c^+ and c^- such that $< F_1, F_2 >$ is positive definite and invariant under w.

15.1. Principal series. Let $\sigma = \tilde{\sigma}_{\eta,ir}$ be the representation of $\bar{S}$ on $\mathcal{F}_{\tilde{\sigma}_{\eta,ir}}$. (See Section 10.) It is easy to see that

$$< f_1, f_2 >= \int f_1(wn(x))\overline{f_2(wn(x))}dx$$

is a convergent $\bar{S}$ invariant inner product on $\mathcal{F}_{\tilde{\sigma}_{\eta,ir}}$. By definition we have

$$< f_1, f_2 >= \int V_{f_1}(x)\overline{V_{f_2}(x)}dx.$$

Since $V_{f_i} \in L^2(\mathbf{R})$ and $\hat{V}_{f_i} \in L^1(\mathbf{R})$ (here $\hat{V} = \hat{V}^1$) we have that

$$\int V_{f_1}(x)\overline{V_{f_2}(x)}dx = \int \hat{V}_{f_1}(x)\overline{\hat{V}_{f_2}(x)}dx. \tag{15.2}$$

Let μ^+ be a character of $\bar{Z}$ extending the central character of $\sigma = \tilde{\sigma}_{\eta,ir}$ and let $\pi = \pi(\mu, \eta, ir) = \mathrm{Ind}(G^*, G, \sigma \otimes \mu)$, realized on a space of functions F: $G^* \times G \to \mathbf{C}$ satisfying (13.1) and the other relations of Section 12. Then by Lemma 12.2, (15.2), (13.6) and (13.7) we have that

$$\begin{aligned}
< F_1, F_2 >&= \int F_1(wn(x), e)\overline{F_2(wn(x), e)}dx + \int F_1(wn(x), e_1)\overline{F_2(wn(x), e_1)}dx \\
&= \int \hat{V}^{\lambda}_{F_1,e}(x)\overline{\hat{V}^{\lambda}_{F_2,e}(x)}dx + \int \hat{V}^{-\lambda}_{F_1,e_1}(x)\overline{\hat{V}^{-\lambda}_{F_2,e_1}(x)}dx \\
&= \int \chi(|x|)^{-1}\overline{\chi(|x|)^{-1}}W^+_{F_1}(t(x))\overline{W^+_{F_2}(t(x))}dx \\
&\quad + \int \chi(|x|)^{-1}\overline{\chi(|x|)^{-1}}W^-_{F_1}(t(x))\overline{W^-_{F_2}(t(x))}dx \\
&= \int W^+_{F_1}(t(x))\overline{W^+_{F_2}(t(x))}d^*x + \int W^-_{F_1}(t(x))\overline{W^-_{F_2}(t(x))}d^*x.
\end{aligned} \tag{15.3}$$

15.2. Complementary series. Let $\sigma = \tilde{\sigma}_{\eta,s}$ with $0 < s < 1/2$. For $f \in \mathcal{F}_\sigma$ we define the intertwining operator

$$(Af)(h) = \int f(w(n(y)h)dy.$$

By (10.5), the integral converges absolutely for every $h \in \bar{S}$ and $Af \in \mathcal{F}_{\tilde{\sigma}_{\eta,s}}$. Therefore, it is clear that the form

$$< f_2, f_1 >= \int (Af_2)(wn(x))\overline{f_1(wn(x))}dx \tag{15.4}$$

is a nonzero $\bar{S}$ invariant hermitian form, hence is a scalar multiple of an $\bar{S}$ invariant inner product. We have

$$\begin{aligned}
(Af)(wn(x)) &= \int f(wn(y)wn(x))dy \\
&= \int \mathrm{sign}(y) f(n(-y^{-1})s(y^{-1})wn(-y^{-1}+x)dy \\
&= \int \gamma(y)\,\overset{\eta+1}{\mathrm{sign}}(y)|y|^{s-1} f(wn(x-y))dy \\
&= \lim_{\delta\to 0}\int \gamma(y)\,\overset{\eta+1}{\mathrm{sign}}(y)|y|^{s-1} V_f(x-y)e^{-\delta|y|}dy \qquad (15.5)\\
&= \lim_{\delta\to 0}\int \gamma(y)\,\overset{\eta+1}{\mathrm{sign}}(y)|y|^{s-1}e^{-\delta|y|}\int \hat{V}_f(z)e^{2\pi i(x-y)z}dz \\
&= \lim_{\delta\to 0}\int \hat{V}_f(z)e^{2\pi ixz}\Big(\int \gamma(y)\,\overset{\eta+1}{\mathrm{sign}}(y)|y|^{s-1}e^{-2\pi iyz-\delta|y|}dy\Big)dz \\
&= \lim_{\delta\to 0}\int \hat{V}_f(z)e^{2\pi ixz} \\
&\quad \Big(\int_0^\infty y^{s-1}e^{(-2\pi iz-\delta)y}dy + \int_{-\infty}^0 i(-1)^{\eta+1}|y|^{s-1}e^{(-2\pi iz-\delta)y}dy\Big)dz.
\end{aligned}$$

Now by [14] (1.5.1),

$$\int_0^\infty y^{s-1}e^{(-2\pi iz-\delta)y}dy = \frac{\Gamma(s)}{(\delta+2\pi iz)^s}$$

and

$$\int_{-\infty}^0 i(-1)^{\eta+1}|y|^{s-1}e^{(-2\pi iz+\delta)y}dy = i(-1)^{\eta+1}\frac{\Gamma(s)}{(\delta-2\pi iz)^s}.$$

Using the dominated convergence theorem we get that

$$\begin{aligned}
(Af)(wn(x)) &= \lim_{\delta\to 0}\int \hat{V}_f(z)e^{2\pi ixz}\Big(\frac{\Gamma(s)}{(\delta+2\pi iz)^s} + i(-1)^{\eta+1}\frac{\Gamma(s)}{(\delta-2\pi iz)^s}\Big)dz \\
&= \int \hat{V}_f(z)e^{2\pi ixz}\Gamma(s)((2\pi iz)^{-s} + i(-1)^{\eta+1}(2\pi iz)^{-s})dz \\
&= \int \hat{V}_f(z)e^{2\pi ixz}(2\pi|z|)^{-s}\Gamma(s)(e^{\mathrm{sign}(z)is\pi/2} \\
&\qquad + i(-1)^{\eta+1}e^{-\mathrm{sign}(z)is\pi/2})dz.
\end{aligned}$$

We let

$$\Delta_{\eta,s}(z) = \Gamma(s)(e^{\mathrm{sign}(z)is\pi/2} + i(-1)^{\eta+1}e^{-\mathrm{sign}(z)is\pi/2}); \tag{15.6}$$

then we have

$$(Af)(wn(x)) = \int \hat{V}_f(z)e^{2\pi ixz}(2\pi|z|)^{-s}\Delta_{\eta,s}(z)dz \tag{15.7}$$

or, more generally,

$$(Af)(wn(x)) = |\lambda|^{1/2}\int \hat{V}_f^\lambda(z)e^{2\pi i\lambda xz}(2\pi|\lambda z|)^{-s}\Delta_{\eta,s}(\lambda z)dz. \tag{15.8}$$

Let μ^+ be a character of $\bar{Z}$ extending the central character of $\sigma = \tilde{\sigma}_{\eta,ir}$ and let $\pi = \pi_{\mu,\eta,ir} = \mathrm{Ind}(G^*, G, \sigma\otimes\mu)$, realized on the space $\mathcal{F}_\pi$ which is a space of functions $F\colon G^*\times G \to \mathbf{C}$ satisfying (13.1). Then by (15.4) and by Lemma 12.2, there exists a scalar β such that

$$\begin{aligned}< F_2, F_1 >=&\beta\int (AF_2)(wn(x),e)\overline{F_1(wn(x),e)}dx\\ &+\beta\int (AF_2)(wn(x),e_1)\overline{F_1(wn(x),e_1)}dx\end{aligned}$$

is a $\bar{G}$ invariant inner product on $\mathcal{F}_\pi$. Here $F_1, F_2 \in \mathcal{F}_\pi$ and

$$(AF)(g^*,g) = \int F(wn(y)g^*,g)dy$$

for every $g^*\in G^*$ and $g\in\bar{G}$. Using (15.8), (13.6) and (13.7) we get

$$\begin{aligned}< F_2, F_1 >=&\beta|\lambda|^{1/2}\int\overline{V_{F_1,e}(x)}\int \hat{V}^\lambda_{F_2,e}(z)e^{2\pi i\lambda xz}(2\pi|\lambda z|)^{-s}\Delta_{\eta,s}(\lambda z)dzdx\\ &+\beta|\lambda|^{1/2}\int\overline{V_{F_1,e_1}(x)}\int \hat{V}^{-\lambda}_{F_2,e_1}(z)e^{-2\pi i\lambda xz}(2\pi|\lambda z|)^{-s}\Delta_{\eta,s}(-\lambda z)dzdx\\ =&\beta\int \hat{V}^\lambda_{F_2,e}(z)\overline{\hat{V}^\lambda_{F_2,e}(z)}(2\pi|\lambda z|)^{-s}\Delta_{\eta,s}(\lambda z)dz\\ &+\beta\int \hat{V}^{-\lambda}_{F_2,e_1}(z)\overline{\hat{V}^{-\lambda}_{F_2,e_1}(z)}(2\pi|\lambda z|)^{-s}\Delta_{\eta,s}(-\lambda z)dz\\ =&\beta(2\pi)^{-s}|\lambda|^{1/2-s}\Delta_{\eta,s}(\lambda)\int W^+_{F_2}(t(z))\overline{W^+_{F_1}(t(z))}d^*z\\ &+\beta(2\pi)^{-s}|\lambda|^{1/2-s}\Delta_{\eta,s}(-\lambda)\int W^-_{F_2}(t(z))\overline{W^-_{F_1}(t(z))}d^*z.\end{aligned} \tag{15.9}$$

To give a choice of β such that the form above is positive definite, it is enough to make sure that the coefficient of $W^+_{F_2}(t(z))\overline{W^+_{F_1}(t(z))}$ is positive. Hence we can choose $\beta = \overline{\Delta_{\eta,s}(\lambda)}$ or $\beta = \Delta_{\eta,s}(\lambda)^{-1}$.

15.3. Discrete series. If π is a holomorphic discrete series and $\lambda > 0$, then $W_F^- = 0$ for every $F \in \mathcal{F}_\pi$. Hence (15.1) reduces to the form

$$< F_1, F_2 >= c^+ \int W_{F_1}^+(t(a))\overline{W_{F_2}^+(t(a))}d^*a. \tag{15.10}$$

It is clear that if the form above is positive definite, then $c^+ > 0$ and we can choose $c^+ = 1$. On the other hand, it follows from Theorem 13.2 that if $W_F^+(t(a)) = 0$ for every $a \in \mathbf{R}^*$, then $W_F^+ = 0$, hence the form above is positive definite. By Remark 15.1 it is enough to show that $<,>$ is invariant under w, that is, $< F_1, wF_2 >=< w^{-1}F_1, F_2 >$ for every $F_1, F_2 \in \mathcal{F}_\pi$ satisfying the above growth conditions. To do that we shall use Theorem 13.2 and Corollary 13.5. We have

$$\begin{aligned} < F_1, wF_2 > &= \int W_{F_1}^+(t(b))\overline{W_{wF_2}^+(t(b))}d^*b \\ &= \int W_{F_1}^+(t(b))\overline{W_{F_2}^+(t(b)w)}d^*b \\ &= \int W_{F_1}^+(t(b))\left(\overline{\int j_{\pi,\lambda}^{+,+}(t(b)wt(a^{-1}))W_{F_2}^+(t(a))d^*a}\right) \\ &= \int \overline{W_{F_2}^+(t(a))}\left(\int j_{\pi,\lambda}^{+,+}(t(a)w^{-1}t(b^{-1}))W_{F_1}^+(t(b))d^*b\right)d^*a \\ &= \int \overline{W_{F_2}^+(t(a))}W_{F_1}^+(t(a)w^{-1})d^*a \\ &=< w^{-1}F_1, F_2 > . \end{aligned} \tag{15.11}$$

Similarly for $\lambda < 0$, where $W_{F_i}^-$ will replace $W_{F_i}^+$ and $j_{\pi,\lambda}^{-,-}$ will replace $j_{\pi,\lambda}^{+,+}$. The anti-holomorphic discrete series will work in the same way. We can restate this result in the following way:

Theorem 15.2: *Let π be a discrete series representation of $\tilde{G}$. Then the form*

$$< F_1, F_2 >= \int W_{F_1}^+(t(a))\overline{W_{F_2}^+(t(a))}d^*a + \int W_{F_1}^-(t(a))\overline{W_{F_2}^-(t(a))}d^*a$$

is a nonzero, $\mathfrak{g}$-skew invariant inner form on the space of K-finite vectors in $\mathcal{F}_\pi$. Hence it is a restriction of a $\tilde{G}$ invariant form to this space. Here $F_1, F_2 \in \mathcal{F}_\pi$, and we note that for each fixed π, one of the integrands in the above formula is identically zero.

16. Inner product for $\overline{SL_2(\mathbf{R})}$

In this section we translate the inner product formulas proved in the previous section for representations of $\bar{S}$ to inner product formulas involving Whittaker functionals. These formulas are needed for future work and will not be used in the rest of this paper.

If $(\pi, \mathcal{F})$ is a representation of $\bar{G}$ with an invariant inner product $<,>$ on $\mathcal{F}$ and W is an $\bar{S}$ invariant subspace of $\bar{G}$, then the restriction of $<,>$ to W gives an invariant inner product on W. In this way we can use the inner product formulas on irreducible representations of $\bar{G}$ to get inner product formulas on irreducible representations of $\bar{S}$. This is only needed for the discrete series since in the other case we already have available inner product formulas.

We fix $\lambda \neq 0$ and let $\psi_\lambda(x) = e^{2\pi i \lambda x}$. If f is a function on G we set

$$L^\lambda(f) = \int f(wn(x))\psi_{-\lambda}(x)dx.$$

16.1. Principal Series. Let $s \in i\mathbf{R}$ and $\eta \in \{0,1\}$. Let $\sigma_{\eta,s}$ be the representation of $\bar{S}$ defined in Definition 10.1 acting on $\mathcal{F}_{\sigma_{\eta,s}}$. If $f \in \mathcal{F}_{\sigma_{\eta,s}}$ then $L^\lambda(f)$ above is defined via analytic continuation or via a principal value integral. In either case we have

$$L^\lambda(s(a)f) = \gamma(a)(\operatorname{sgn}(a))^\eta |a|^{1-s} L^{a^2\lambda}(f).$$

Let $f_1, f_2 \in \mathcal{F}_{\sigma_{\eta,s}}$. By (15.2) the following is an $\bar{S}$ invariant inner product:

$$\begin{aligned} < f_1, f_2 > &= \int L^\lambda(f_1)\overline{L^\lambda(f_2)}d\lambda \\ &= \int |\lambda a| L^{\lambda a^2}(f_1)\overline{L^{\lambda a^2}(f_2)}da + \int |\lambda a| L^{-\lambda a^2}(f_1)\overline{L^{-\lambda a^2}(f_2)}da \qquad (16.1) \\ &= |\lambda| \int L^\lambda(s(a)f_1)\overline{L^\lambda(s(a)f_2)}d^*a + |\lambda| \int L^{-\lambda}(s(a)f_1)\overline{L^{-\lambda}(s(a)f_2)}d^*a. \end{aligned}$$

16.2. Complementary Series. Let $s \in \mathbf{R}$ and $\eta \in \{0,1\}$. Let $\mathcal{F}_{\sigma_{\eta,s}}$ be the appropriate representation space. Then using (15.4), (15.7) and similar arguments as in the principal series case, we get that the following is an $\bar{S}$ invariant inner product on $\mathcal{F}_{\sigma_{\eta,s}}$:

$$\begin{aligned} < f_1, f_2 >= &\int L^\lambda(s(a)f_1)\overline{L^\lambda(s(a)f_2)}d^*a \\ &+ \frac{\Delta_{\eta,s}(-\lambda)}{\Delta_{\eta,s}(\lambda)} \int L^{-\lambda}(s(a)f_1)\overline{L^{-\lambda}(s(a)f_2)}d^*a \end{aligned} \qquad (16.2)$$

where $\Delta_{\eta,s}(\lambda)$ is defined in (15.6).

16.3. Discrete Series. Let $s \in 1/2 + \mathbf{N}$ and $\eta \in \{0,1\}$. Let $\mathcal{F}_{\tilde{\sigma}_{\eta,s}}$ be the appropriate representation space. Then using Theorem 15.2 and the vanishing of certain Whittaker functionals, we get that the following is an $\tilde{S}$ invariant inner product on the space of K-finite vectors in $\mathcal{F}_{\tilde{\sigma}_{\eta,s}}$:

$$(16.3) \qquad < f_1, f_2 >= \int L^\lambda(s(a)f_1)\overline{L^\lambda(s(a)f_2)}d^*a.$$

Here, we need to take $\lambda > 0$ if $\sigma_{\eta,s}$ is a holomorphic discrete series representation and $\lambda < 0$ if $\sigma_{\eta,s}$ is an anti-holomorphic discrete series representation.

17. Bessel distributions on $\overline{GL_2(\mathbf{R})}$

Let (π, H) be a genuine, irreducible, unitary representation of $\bar{G}$ on a Hilbert space H and let $< v, w >$ be a nonzero $\bar{G}$ invariant inner product on H. For every continuous functional $L : H_\infty \to \mathbf{C}$ and every $f \in C_c^\infty(G)$ there exists a unique vector $v_{f,L} \in H_\infty$ such that

$$(17.1) \qquad L(\pi(f)w) =< w, v_{f,L} > \quad \text{for every } w \in H.$$

Let μ_π be the central character of π and let μ^+ and μ^- be extensions of μ_π to $\bar{Z}$ as defined in (13.3). Fix $\psi = \psi_\lambda$ a nontrivial character of N. Let L^+ and L^- be (ψ, μ^+) and (ψ, μ^-) Whittaker functionals respectively. We assume that they are nonzero if possible. By (15.1) we can normalize L^+ and L^- so that

$$< v, w >= \int W_v^+(t(a))\overline{W_w^+(t(a))}d^*a + \int W_v^-(t(a))\overline{W_w^-(t(a))}d^*a.$$

Here $W_v^+(\bar{g}) = L^+(\pi(\bar{g})v)$. We define the distributions $J_{\pi,\psi}^+$ and $J_{\pi,\psi}^-$ on $C_c^\infty(\bar{G})$ by

$$(17.2) \qquad J_{\pi,\psi}^+(f) = \overline{L^+(v_{f,L^+})}, \quad J_{\pi,\psi}^-(f) = \overline{L^-(v_{f,L^-})}.$$

Lemma 17.1: *Let $u \in H_\infty$ be such that $W_u^-(t(a)) = 0$ for all $a \in \mathbf{R}^*$. If π is a discrete series representation then we assume that $W_u^+(t(a))$ has a high order of vanishing at $a = 0$. Then*

$$\int J_{\pi,\psi}^+(\rho_r(t(a))f)W_u^+(t(a))d^*a = \int_{\bar{G}} f(g)W_u^+(\bar{g})d\bar{g}.$$

The proof is the same as the proof of Lemma 7.1. It is clear that we can replace μ^+ with μ^- in the above lemma. We define the distributions

$$(17.3) \qquad \tilde{J}_{\pi,\psi}^+(f) = \int j_{\pi,\psi}^{+,+}(\bar{g})f(\bar{g})d\bar{g}, \quad \tilde{J}_{\pi,\psi}^-(f) = \int j_{\pi,\psi}^{-,-}(\bar{g})f(\bar{g})d\bar{g}.$$

LEMMA 17.2: *Let $u \in H_\infty$ be such that $W_u^-(t(a)) = 0$ for all $a \in \mathbf{R}^*$ and such that $W_u^+(t(a))$ has a high order of vanishing at $a = 0$. Then*

$$\int \tilde{J}^+_{\pi,\psi}(\rho_r(t(a))f)W_u^+(t(a))d^*a = \int_{\bar{G}} f(g)W_u^+(\bar{g})d\bar{g}.$$

The proof is the same as the proof of Lemma 7.2. We replace (6.2) in the proof of Lemma 7.2 with Theorem 13.2.

COROLLARY 17.3: *Let $f \in C_c^\infty(G)$. Then*

$$J^+_{\pi,\psi}(f) = \tilde{J}^+_{\pi,\psi}(f), \quad J^-_{\pi,\psi}(f) = \tilde{J}^-_{\pi,\psi}(f).$$

Proof: The proof is the same as the proof of Corollary 7.3 using Lemma 17.1 and Lemma 17.2. ∎

18. Bessel distributions on $\overline{SL_2(\mathbf{R})}$

In this section we show that the Bessel distributions on $\bar{S} = \overline{SL_2(\mathbf{R})}$ are given by Bessel functions which are restrictions of Bessel functions from $\bar{G} = \overline{GL_2(\mathbf{R})}$.

Let (σ, W) be a genuine, irreducible, unitary, representation of $\bar{S}$ on a Hilbert space W. We assume σ has a nontrivial $\psi = \psi_\lambda$ Whittaker functional L. We note that any σ has a nontrivial ψ_λ Whittaker functional for some $\lambda \in \mathbf{R}^*$.

Let μ be the unique character on $\bar{Z}$ which is trivial on Z^2 and agrees with the central character of σ on the center of $\bar{S}$. Let $\pi = \mathrm{Ind}(\mu \times \sigma, \bar{G}^*, \bar{G})$ be the representation of $\bar{G}$ associated to σ as in Section 12. Let $\mathcal{F}_\pi$ be the space of π.

We can write $\mathcal{F} = \mathcal{F}_\pi = \mathcal{F}_\pi^+ \oplus \mathcal{F}_\pi^-$ where

$$\mathcal{F}_\pi^+ = \{v \in \mathcal{F}_\pi : \pi(\bar{z}v) = \mu^+(\bar{z})v, \quad \text{for all } z \in \bar{Z}\},$$

and $\mathcal{F}_\pi^-$ is defined similarly. $\bar{S}$ stabilizes $\mathcal{F}_\pi^+$ and we can identify $\mathcal{F}_\pi^+$ with W. We also identify W_∞ with $\mathcal{F}_\infty^+$, which is the subspace of smooth vectors in $\mathcal{F}_\pi^+$. We also have that

$$\mathcal{F}_\infty = \mathcal{F}_\infty^+ \oplus \mathcal{F}_\infty^-.$$

Given our ψ Whittaker function L on $\mathcal{F}_\infty$, we define a (ψ, μ) Whittaker functional L^+ on $\mathcal{F}_\infty$ by setting $L^+ = L$ on $\mathcal{F}_\infty^+$ and $L^+ = 0$ on $\mathcal{F}_\infty^-$.

Let $< v, w >$ be a $\bar{S}$ invariant inner product on W. We can extend $<,>$ to a $\bar{G}$ invariant inner product on $\mathcal{F}_\pi$ so that $<,>$ is the same on $\mathcal{F}_\infty^+$ and so that $\mathcal{F}^+$ is orthogonal to $\mathcal{F}^-$. We normalize L^+ (by normalizing L) so that (15.1) holds with $c^+ = 1$. For each function $\phi \in C_c^\infty(\bar{S})$ there exists a vector $v_{\phi,L} \in \mathcal{F}_\infty^+$ such that

$$L(\sigma(\phi)w) = < w, v_{\phi,L} > \quad \text{for all } w \in \mathcal{F}^+.$$

We define the distribution $J_{\sigma,\psi}$ on $C_c^\infty(\bar{S})$ by

$$J_{\sigma,\psi}(\phi) = \overline{L(v_{\phi,L})}.$$

THEOREM 18.1:

$$J_{\sigma,\psi}(\phi) = \int_{\bar{S}} j_{\pi,\psi}^{+,+}(\bar{s})\phi(\bar{s})d\bar{s}$$

where π is the induced representation of $\bar{G}$ defined above.

Proof: Fix $\phi \in C_c^\infty(\bar{S})$. Let $\bar{Z}^2 = Z^2 \times \pm 1$ be the center of $\bar{G}$ and let $U \subset Z^2 \subset \bar{Z}^2$ be a small open set around the identity element $[e,1]$ such that the mapping $(\bar{s}, z) \mapsto \bar{s}z$ from $\bar{S} \times U \to \bar{G}$ is an injection onto an open set in $\bar{G}$. Let $\alpha \in C_c^\infty(U)$ be such that $\int_{Z^2} \alpha(z)dz = 1$. We define $\Phi \in C_c^\infty(\bar{G})$ by

$$\Phi(\bar{g}) = \begin{cases} \phi(\bar{s})\alpha(\bar{z}) & \text{if } \bar{g} = \bar{s}z \text{ for } \bar{s} \in \bar{S} \text{ and } z \in U; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\Phi \in C_c^\infty(\bar{G})$. Since μ is trivial on Z^2, it follows that for every $w \in \mathcal{F}^+$

$$\pi(\Phi)w = \int_{\bar{G}} \Phi(\bar{g})\pi(\bar{g})wd\bar{g} = \int_{\bar{S}}\int_{Z^2} \phi(\bar{s})\alpha(z)\sigma(\bar{s})\mu(z)wd\bar{s}dz = \sigma(\phi)w.$$

It follows that $v_{L,\phi} = v_{L^+,\Phi}$, hence $J_{\sigma,\psi}(\phi) = J_{\pi,\psi}^+(\Phi)$. By Corollary 17.3,

$$\begin{aligned} J_{\pi,\psi}^+(\Phi) &= \int_{\bar{G}} j_{\pi,\psi}^{+,+}(\bar{g})\phi(\bar{g})d\bar{g} = \int_{\bar{S}} j_{\pi,\psi}^{+,+}(\bar{s})\phi(\bar{s})d\bar{s}\int_{Z^2} \alpha(z)\mu(z)dz \\ &= \int_{\bar{S}} j_{\pi,\psi}^{+,+}(\bar{s})\phi(\bar{s})d\bar{s}. \qquad \blacksquare \end{aligned}$$

We now define

$$j_{\sigma,\psi}(\bar{s}) = j_{\sigma,\psi}(\bar{s}) = j_{\sigma,\lambda}(\bar{s}) = j_{\pi,\lambda}^{+,+}(\bar{s})$$

where $\pi = \mathrm{Ind}(\mu \times \sigma, \bar{G}^*, \bar{G})$. For $f' \in C_c^\infty(\bar{S})$, we define as in [11] the orbital integral

$$O_{f',\psi}^{N,N}(g) = \int\int \phi(n(x)gn(y))\psi(x)\psi(y)dxdy. \tag{18.1}$$

COROLLARY 18.2: *Let σ be a genuine irreducible unitary representation of $\bar{S}$ with a nonzero ψ Whittaker functional. Then*

$$J_{\sigma,\psi}(f') = \int j_{\sigma,\psi}(ws(a))O_{f',\psi}^{N,N}(ws(a))|a|^2d^*a.$$

We conclude this section by listing the values of $j_{\sigma,\lambda}(ws(a))$ for the various irreducible unitary representations of $\bar{S}$.

BESSEL FUNCTIONS ON $\bar{S}$. Let $\psi = \psi_\lambda$.

(1) $\sigma = \sigma_{\eta,ir}$, $r \in \mathbf{R}$.

$$j_{\sigma,\psi}(ws(a)) = \begin{cases} |\lambda|^{1/2}\gamma(a)(\mathrm{sign}(a))^\eta |a|^{-1}\frac{-\pi}{\sin(\pi ir)}[(e^{-(\pi r)/2} + i(-1)^{\eta+1}e^{(\pi r)/2})J_{ir}(4\pi|\lambda||a|^{-1}) \\ +(-e^{(\pi r)/2} + i(-1)^\eta e^{-(\pi r)/2})J_{-ir}(-4\pi|\lambda||a|^{-1})] \\ \text{if } \lambda > 0; \\ |\lambda|^{1/2}\gamma(a)(\mathrm{sign}(a))^\eta |a|^{-1}\frac{-\pi}{\sin(\pi ir)}[(e^{(\pi r)/2} + i(-1)^{\eta+1}e^{-(\pi r)/2})J_{ir}(4\pi|\lambda||a|^{-1}) \\ +(-e^{-(\pi r)/2} + i(-1)^\eta e^{(\pi r)/2})J_{-ir}(-4\pi|\lambda||a|^{-1})] \\ \text{if } \lambda < 0; \end{cases}$$

(2) $\sigma = \sigma_{\eta,s}$, $s \in \mathbf{R}$, $0 < s < 1/2$.

$$j_{\sigma,\psi}(ws(a)) = \begin{cases} |\lambda|^{1/2}\gamma(a)(\mathrm{sign}(a))^\eta |a|^{-1}\frac{-\pi}{\sin(\pi s)}[(e^{(\pi is)/2} + i(-1)^{\eta+1}e^{-(\pi is)/2})J_{s}(4\pi|\lambda||a|^{-1}) \\ +(-e^{-(\pi is)/2} + i(-1)^\eta e^{(\pi is)/2})J_{-s}(-4\pi|\lambda||a|^{-1})] \\ \text{if } \lambda > 0; \\ |\lambda|^{1/2}\gamma(a)(\mathrm{sign}(a))^\eta |a|^{-1}\frac{-\pi}{\sin(\pi s)}[(e^{-(\pi is)/2} + i(-1)^{\eta+1}e^{(\pi is)/2})J_{s}(4\pi|\lambda||a|^{-1}) \\ +(-e^{(\pi is)/2} + i(-1)^\eta e^{-(\pi is)/2})J_{-s}(-4\pi|\lambda||a|^{-1})] \\ \text{if } \lambda < 0; \end{cases}$$

(3) $\sigma = \tilde{\sigma}_{\eta,s}$, $s \in -1/2 + \mathbf{N}$, $s \in -\nu + 2\mathbf{N}$.

$$j_{\sigma,\psi}(ws(a)) = \begin{cases} 2|\lambda|^{1/2}\gamma(a)(\mathrm{sign}(a))^\eta |a|^{-1}\frac{\pi}{\sin \pi s}e^{-3\pi is/2}J_s(4\pi|\lambda||a|^{-1}) & \text{if } \lambda > 0; \\ 0 & \text{otherwise.} \end{cases} \tag{18.2}$$

(4) $\sigma = \tilde{\sigma}_{\eta,s}$, $s \in -1/2 + \mathbf{N}$, $s \in \nu + 2\mathbf{N}$.

$$j_{\sigma,\psi}(ws(a)) = \begin{cases} -2|\lambda|^{1/2}\gamma(a)(\mathrm{sign}(a))^\eta |a|^{-1}\frac{\pi}{\sin \pi s}e^{3\pi is/2}J_s(4\pi|\lambda||a|^{-1}) & \text{if } \lambda < 0; \\ 0 & \text{otherwise.} \end{cases} \tag{18.3}$$

19. Bessel identities

In this section we establish a correspondence between irreducible unitary representations of $G = PGL_2(\mathbf{R})$ and $\bar{S} = \overline{SL_2(\mathbf{R})}$ by deriving identities between their Bessel functions. This correspondence is in fact the Waldspurger correspondence which was established in [21] using theta correspondence techniques.

Fix an additive character $\psi = \psi_\lambda$ of $\mathbf{R}$ given by $\psi(x) = e^{2\pi i\lambda x}$ with $\lambda \neq 0$. Let $D \in \mathbf{R}^*$ and let $\psi^D(x) = \psi(Dx)$. Let $\gamma(x, \psi^D)$ be the Weil factor defined by

$$\gamma(x, \psi^D) = (2D)^{-1/2}e^{\mathrm{sgn}(\lambda Dx)\pi i/4}.$$

Let σ be an irreducible unitary representation of $\bar{S}$ which is not a Weil representation r_ψ^+. If σ has nontrivial ψ^D-Whittaker functional, then its Bessel function j_{σ,ψ^D} is defined as in Theorem 18.1 and the discussion below it. We change the normalization of j_{σ,ψ^D} by changing the normalization of ds. We will fix $ds = |a|^{-2} d_\lambda x d_\lambda^* a d_\lambda y$ on the set of elements of the form $n(x)s(a)wn(y)$. This will introduce an extra factor of $|D|^{-1/2}$ to the formulas in Section 18 where we replace $\psi = \psi_\lambda$ with $\psi^D = \psi_{\lambda D}$. Define a *transfer factor*

$$\Delta_{D,\psi}(x) = \gamma(x, \psi^D)\psi(2D/x)|x|^{1/2}. \tag{19.1}$$

Let π be an irreducible unitary representation of G.

Definition 19.1: We say that an irreducible genuine unitary representation σ of $\bar{S}$ corresponds to π if the following equality (Bessel identity) holds:

$$i_{\pi,\psi}(n(x/4D)w_0) = \frac{\Delta_{D,\psi}(x)\epsilon(\pi, 1/2, \psi)|2D|}{L(\pi, 1/2)} j_{\sigma,\psi^D}(ws(x)) \tag{19.2}$$

for all $x \in k^*$

The following theorem is the main theorem of this paper. Let $\pi_{\eta,s}$ be the representation (or irreducible subrepresentation) of G induced from the character (μ, μ^{-1}) of the Borel B with $\mu(x) = |x|^s(\mathrm{sgn}(x))^\eta$ and $\mathrm{Re}(s) \geq 0, \eta \in \{0, 1\}$. Let $\sigma_{\eta,s}$ be the representation (or irreducible subrepresentation) of $\bar{S}$ induced from the character μ' of the Borel $\bar{B}_S$ with $\mu'([x, \epsilon]) = \epsilon\gamma(x)(\mathrm{sgn}(x))^\eta |x|^s (\mathrm{sgn}(x))^\eta$ and $\mathrm{Re}(s) \geq 0, \eta \in \{0, 1\}$. (See Section 10.)

THEOREM 19.2: *For each irreducible admissible unitary representation π of G, there exists a corresponding representation π' of $\bar{S}$ satisfying the Bessel identity (19.2). The following diagram describes the correspondence:*

G	$\bar{S}$	
$\pi_{\eta,s}$	$\sigma_{(2\eta-1+\mathrm{sgn}(\lambda D))/2,s}$	*if* $s \in i\mathbf{R}$ *or* $0 < s < 1/2$;
$\pi_{\eta,d-1/2}$	$\sigma_{(1+\mathrm{sgn}(\lambda D)(-1)^{d+1})/2,d-1/2}$	*if* $d \in \mathbf{N}$.

Remark 19.3: We took the approach of using the Bessel identity to define the correspondence. It is easy to check that $\sigma = \Theta(\pi, \psi^D)$, where $\Theta(*, \psi^D)$ is the theta correspondence used in [21]. Thus the Theorem is equivalent to: if $\pi' = \Theta(\pi, \psi^D)$, then (19.2) holds.

Proof: The proof is a simple comparison between the formulas in Corollary 8.4 and the formulas at the end of Section 18. To verify the equality, we note that by [9] p. 2.27 (122), if $\pi = \pi_{\eta,s}$ with $s \in i\mathbf{R}$ or $0 < s < 1/2$ then $\epsilon(\pi, 1/2, \psi) =$

$(-1)^\eta$; and by [9] p. 2.29 (133), if $\pi = \pi_{1,d-1/2}$ with $d \in \mathbf{N}$ then $\epsilon(\pi, 1/2, \psi) = (-1)^d$. ∎

We can now prove an identity in the level of distributions. By [11], for each $f \in C_c^\infty(G)$ there exists $f' \in C_c^\infty(\bar{S})$ such that

$$O_{f,\psi}^{A,N}(n(a/4D)w_0) = O_{f',\psi^D}^{N,N}(ws(a))\psi(-2D/a)|a|^{1/2}\gamma(a,\psi^D)^{-1} \tag{19.3}$$

for all $a \in \mathbf{R}^*$. (See (3.24) and (18.1) for the definitions of these orbital integrals.) Hence using Theorem 5.3, (3.25), Corollary 18.2 and Theorem 19.2 we get:

THEOREM 19.4: *Assume that f and f' satisfy (19.3) and that π and σ correspond as in Theorem 19.2. Then*

$$I_{\pi,\psi}(f) = J_{\sigma,\psi^D}(f')\epsilon(\pi, 1/2, \psi)|2D|^{1/2}/L(\pi, 1/2). \tag{19.4}$$

20. Appendix 1: Bounds on classical K-Bessel functions

We will need some bounds on classical K-Bessel functions. These bounds will follow from the following integral representation ([14], p. 140): Assume $\mathrm{Re}(\nu) > -1/2$ and $|\arg(z)| < \pi$. Then

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} \frac{e^{-z}}{\Gamma(\nu + 1/2)} \int_0^\infty e^{-s} s^{\nu-1/2}\left(1 + \frac{s}{2z}\right)^{\nu-1/2} ds. \tag{20.1}$$

LEMMA 20.1: *Assume that ν is in a compact set Q in the region $0 \leq \mathrm{Re}(\nu) \leq 1/2$ and that $z \neq 0$, $\mathrm{Re}(z) \geq 0$. Then*

$$|K_\nu(z)| <<_Q |z|^{-1/2} e^{-\mathrm{Re}(z)}.$$

Proof: By taking the absolute value in the integral representation (20.1) we get

$$|K_\nu(z)| << |z|^{-1/2} \frac{e^{-\mathrm{Re}(z)}}{|\Gamma(\nu + 1/2)|} \int_0^\infty e^{-s} s^{\mathrm{Re}(\nu)-1/2} \left|\left(1 + \frac{s}{2z}\right)^{\nu-1/2}\right| ds.$$

Since

$$|w^{\nu-1/2}| = |w|^{\mathrm{Re}(\nu)-1/2} e^{-\arg(w)\,\mathrm{Im}(\nu)}$$

(here $|\arg(w)| < \pi$) and since ν is in a compact set Q as above, we get that

$$|w^{\nu-1/2}| <<_Q |w|^{\mathrm{Re}(\nu)-1/2}.$$

We set $w = 1 + s/2z$. Since $\mathrm{Re}(z) \geq 0$, we get that $|w| \geq 1$. Since $\mathrm{Re}(\nu) \leq 1/2$, we get $|w^{\nu-1/2}| <<_Q 1$. Hence

$$\begin{aligned}|K_\nu(z)| &<<_Q |z|^{-1/2} e^{-\mathrm{Re}(z)} \int_0^\infty e^{-s} s^{\mathrm{Re}(\nu)-1/2} ds \\ &= |z|^{-1/2} e^{-\mathrm{Re}(z)} \Gamma(\mathrm{Re}(\nu) + 1/2)\end{aligned}$$

and the result follows. (We used the fact that both

$$1/|\Gamma(\nu + 1/2)| \quad \text{and} \quad \Gamma(\mathrm{Re}(\nu) + 1/2)$$

are bounded on Q.) ∎

We can improve the above result as follows:

LEMMA 20.2: *Assume that ν is in a compact set Q in the region $0 \leq \mathrm{Re}(\nu) \leq 1/2$ and that $z \neq 0$, $\mathrm{Re}(z) \geq 0$. Assume that $\lambda = \max\{\mathrm{Re}(\nu) : \nu \in Q\} \neq 0$. Then*

$$|K_\nu(z)| <<_Q |z|^{-\lambda} e^{-\mathrm{Re}(z)}.$$

Proof: The proof follows the same lines as above. We replace the inequality $|(1 + s/2z)^{\nu-1/2}| <<_Q 1$ with the inequality

$$|(1 + s/2z)^{\nu-1/2}| <<_Q |s/2z|^{\lambda-1/2}.$$

To get the last inequality, we note that $z \neq 0$, $\mathrm{Re}(z) \geq 0$, and $s > 0$ imply that $|1 + s/2z| > 1$. Hence $|1 + s/2z|^{\mathrm{Re}(\nu)-1/2} \leq |1 + s/2z|^{\lambda-1/2}$. Also, $|1 + s/2z| > |s/2z|$, hence $|1 + s/2z|^{\lambda-1/2}| \leq |s/2z|^{\lambda-1/2}$. ∎

LEMMA 20.3: *Assume that $\mathrm{Re}(\nu) > 1/2$ and that $|\arg(z)| < \pi$. Then there exist positive constants C_1, C_2 which depend on ν (but not on z) such that*

$$|K_\nu(z)| \leq C_1 |z|^{-1/2} e^{-\mathrm{Re}(z)} + C_2 |z|^{-\mathrm{Re}(\nu)} e^{-\mathrm{Re}(z)}.$$

Proof: We again use the integral representation (20.1). We will divide the integration into two regions: From 0 to $2|z|$ and from $2|z|$ to ∞. When $s \leq 2|z|$ we have that $|1 + s/2z| \leq 2$. When $s > 2|z|$ we have that $|1 + s/2z| \leq 2|s/2z| = |s/z|$. Hence

$$\begin{aligned}|K_\nu(z)| &<<_\nu |z|^{-1/2} e^{-\mathrm{Re}(z)} \\ &\times \left(\int_0^{2|z|} e^{-s} s^{\mathrm{Re}(\nu)-1/2} 2^{\mathrm{Re}(\nu)-1/2} ds + \int_{2|z|}^\infty e^{-s} s^{\mathrm{Re}(\nu)-1/2} |s/z|^{\mathrm{Re}(\nu)-1/2} ds \right) \\ &<<_\nu |z|^{-1/2} e^{-\mathrm{Re}(z)} \\ &\times \left(\int_0^\infty e^{-s} s^{\mathrm{Re}(\nu)-1/2} 2^{\mathrm{Re}(\nu)-1/2} ds + |z|^{1/2-\mathrm{Re}(\nu)} \int_0^\infty e^{-s} s^{2\,\mathrm{Re}(\nu)-1} ds \right) \\ &<<_\nu C_{1,\nu} |z|^{-1/2} e^{-\mathrm{Re}(z)} + C_{2,\nu} |z|^{-\mathrm{Re}(\nu)} e^{-\mathrm{Re}(z)}.\end{aligned}$$

∎

From Lemma 20.1 and Lemma 20.3 we get:

COROLLARY 20.4: *Fix ν such that* $\mathrm{Re}(\nu) \geq 0$. *Assume that* $\mathrm{Re}(z) \geq 0$ *and that* $|z|$ *is large. Then*

$$|K_\nu(z)| << |z|^{-1/2}.$$

21. Appendix 2: Computation of Bessel functions for $GL_2(\mathbf{R})$

In this appendix we give a proof of the kernel formula (6.1) of Cogdell and Piatetski-Shapiro [5]. Our proof is different from the one which was communicated to us by Jim Cogdell. The proof is identical to the proof of the kernel formula (13.8) for $\tilde{G}$ which was derived in Section 10, Section 11 and Section 13. A different proof for this kernel formula in the case of principal series representations was obtained independently by Motohashi in [15].

We will sketch the details here and refer the reader to the proofs in the above sections. Let $G = GL_2(\mathbf{R})$. Let $\eta \in \{0,1\}$ and $s \in \mathbf{C}$. Let $\Pi_{\eta,s}$ be the representation of G on the space of smooth functions $f\colon G \to \mathbf{C}$ satisfying

$$f(n(x)t(a)z(b)h) = (\mathrm{sign}(a))^\eta |a|^{s+1/2} f(h) \tag{21.1}$$

where $x \in \mathbf{R}$, $a \in \mathbf{R}^*$ and $h \in G$.

We shall assume that $\mathrm{Re}(s) \geq 0$. When $\Pi_{\eta,s}$ is irreducible, that is, $s \neq d - 1/2$ for some positive integer d, we let $\pi_{\eta,s} = \Pi_{\eta,s}$. When $s = d - 1/2$ and d is a positive integer, then $\Pi_{\eta,s}$ is reducible and we let $\pi_{\eta,s} = \pi_{\eta,d-1/2}$ be the unique irreducible subspace of $\Pi_{\eta,s}$. Since $\pi_{0,d-1/2} \simeq \pi_{1,d-1/2}$, it is enough for us to consider the case $\pi_{1,d-1/2}$. We will only be interested in the unitary representations: The complementary series representations where $0 < s < 1/2$, the principal series representations with $\mathrm{Re}(s) = 0$ and the discrete series representations $\pi_d = \pi_{1,d-1/2}$. It is possible to obtain a kernel formula for all irreducible admissible representations of G, but in the nonunitary case the kernel formula can only be applied to certain vectors in the representations whose Whittaker functions behave nicely at zero. It will be obvious from our proofs that such formulas hold in these cases. In the unitary representations case, the kernel formula should hold for every smooth vector. We will prove that the kernel formula holds for every smooth vector in the complementary series and in the principal series and that it holds for every K-finite vector in the discrete series.

We let $\mathcal{V}(\eta, s)$ be the space of functions V_f, $f \in \pi_{\eta,s}$ defined by

$$V_f(x) = f(w_0 n(x)).$$

As in (10.5), we have that $V_f(x)$ and all its derivatives satisfy

$$V_f^{(n)}(x) = O(|x|^{-2s-1}).$$

Hence, if $\mathrm{Re}(s) > 0$ and $V \in \mathcal{V}(\eta, s)$ then we can define

$$\hat{V}(y) = \hat{V}^1(y) = \int_{\mathbf{R}} V(x)e^{-2\pi i y x}dx. \tag{21.2}$$

It follows from standard Fourier analysis that:

LEMMA 21.1: *Assume that* $\mathrm{Re}(s) > 0$ *and* $V \in \mathcal{V}(\eta, s)$. *Then* $\hat{V}$ *is continuous, bounded and rapidly decreasing.*

Finally, we shall consider another model for $\pi_{\eta,s}$, which we denote by $\hat{\mathcal{V}}(\eta, s)$, consisting of functions $\hat{V}$, $V \in \mathcal{V}(\eta, s)$. If $\mathrm{Re}(s) > 0$ then $\hat{V}$ is defined via (21.2). If $\mathrm{Re}(s) = 0$ then $\hat{V}$ is defined via analytic continuation of a section in the following way: Let $K = SO(2)$. A function $f_s \in \Pi_{\eta,s}$ is a section if the restriction of f_s to K is a smooth function independent of s. For every $\phi \in \Pi_{\eta,ir_0}$ we can choose a section $f_s \in \Pi_{\eta,s}$ such that $f_{ir_0} = \phi$. We denote $V_s = V_{f_s}$, hence $V_{ir_0} = V_\phi$. Then

$$\hat{V}_\phi(y) = \lim_{s\to ir_0} \hat{V}_s(y) = \lim_{s\to ir_0} \int_{\mathbf{R}} V_s(x)e^{-2\pi i y x}dx \tag{21.3}$$

where the limit is taken on a compact path from the right of ir_0. Our aim is to find an explicit formula of the following nature:

$$w_0\hat{V}(b) = (w_0V)^\wedge(b) = \int_{\mathbf{R}} k_{s,\eta}(a, b)\hat{V}(a)d_1a. \tag{21.4}$$

To do that we shall need the following Theorem. For the proof see Theorem 10.3.

THEOREM 21.2: *Let* $f_s \in \Pi_{\eta,s}$ *be a section and let* s *be in a compact set* Q *in the half plane* $0 \le \mathrm{Re}(s) \le 1/4$. *Let* $V_s = V_{f_s} \in \mathcal{V}(\eta, s)$ *be a section as above. Then* $|\hat{V}_s(y)| <<_Q |y|^{-1/4}$ *and* $\hat{V}_s$ *is rapidly decreasing, independent of* $s \in Q$.

We shall also need the following Lemma for the discrete series. For the proof see Lemma 11.7.

LEMMA 21.3: *Let* $V \in \tilde{V}(1, d - 1/2)$ *be a K-finite vector. Then* $|\hat{V}(a)| = O(|a|^{2d-1})$ *when* $|a| > 0$ *is small.*

21.1. Complementary series. Fix $0 < s < 1/2$. Let $V \in \mathcal{V}(\eta, s)$. As in (11.4) we use Fourier inversion to write

$$\hat{V}(y) = \lim_{\delta \to 0^+} \int V(x) e^{-2\pi i y x} e^{-\delta(|x|+|x|^{-1})} dx.$$

Now we get

$$\begin{aligned}
(w_0 \hat{V})(b) &= \lim_{\delta \to 0^+} \int f(w_0 n(x) w_0) e^{-2\pi i x b} e^{-\delta(|x|+|x|^{-1})} dx \\
&= \lim_{\delta \to 0^+} \int f(n(x^{-1}) t(-x^{-2}) z(x) w_0 n(x^{-1})) e^{-2\pi i x b} e^{-\delta(|x|+|x|^{-1})} da.dx \\
&= \lim_{\delta \to 0^+} \int (-1)^\eta |x|^{-2s-1} e^{-2\pi i x b} e^{-\delta(|x|+|x|^{-1})} V(x^{-1}) dx \qquad (21.5) \\
&= \lim_{\delta \to 0^+} (-1)^\eta \int |x|^{-2s-1} e^{-2\pi i x b} e^{-\delta(|x|+|x|^{-1})} \left(\int \hat{V}(a) e^{2\pi i a/x} \right) dx \\
&= \lim_{\delta \to 0^+} \int \hat{V}(a) \left((-1)^\eta \int |x|^{-2s-1} e^{-2\pi i x b} e^{2\pi i a/x} e^{-\delta(|x|+|x|^{-1})} dx \right) da.
\end{aligned}$$

Here, the change of order is justified as in Lemma 11.2. Let

$$k_{s,\eta,\delta}(a,b) = (-1)^\eta \int |x|^{-2s-1} e^{-2\pi i x b} e^{2\pi i a/x} e^{-\delta(|x|+|x|^{-1})} dx.$$

Then for $\mathrm{Re}(s) > 0$ and $V \in \mathcal{V}(\eta, s)$ we have

$$w\hat{V}(b) = \lim_{\delta \to 0^+} \int \hat{V}(a) k_{s,\eta,\delta}(a,b) da. \qquad (21.6)$$

By (11.9) we have

$$k_{s,\eta,\delta}(a,b) = (-1)^\eta (K(2s, \delta - 2\pi i a, \delta + 2\pi i b) + K(2s, \delta + 2\pi i a, \delta - 2\pi i b).$$

Now we use Lemma 21.1, Lemma 11.4, Lemma 20.2 and Corollary 20.4 as in Section 11.3 (see also the proof of Lemma 11.5) to apply the dominated convergence in (21.6). We conclude that if $\hat{V}(a)$ has a large order of vanishing at $a = 0$, or if V is a smooth vector in the complementary series, or V is a K-finite vector in the discrete series, then

$$w\hat{V}(b) = \lim_{\delta \to 0^+} \int \hat{V}(a) \big(\lim_{\delta \to 0^+} k_{s,\eta,\delta}(a,b) \big) da. \qquad (21.7)$$

Let $k_{s,\eta}(a,b) = \lim_{\delta \to 0^+} k_{s,\eta,\delta}(a,b)$. Then

$$\begin{aligned}
k_{s,\eta}(a,b) =& (-1)^\eta (K(2s, -2\pi i a, +2\pi i b) + K(2s, +2\pi i a, \delta - 2\pi i b)) \\
=& (-1)^\eta 2 |a/b|^{-s} [e^{\frac{(-1-2s)(-\operatorname{sgn}(a) - \operatorname{sgn}(b))\pi i}{4}} K_{2s}(4\pi \sqrt{|ab|} e^{\frac{(-\operatorname{sgn}(a) + \operatorname{sgn}(b))\pi i}{4}}) \qquad (21.8) \\
&+ e^{\frac{(-1-2s)(\operatorname{sgn}(a) + \operatorname{sgn}(b))\pi i}{4}} K_{2s}(4\pi \sqrt{|ab|} e^{\frac{(\operatorname{sgn}(a) - \operatorname{sgn}(b))\pi i}{4}})].
\end{aligned}$$

21.2. PRINCIPAL SERIES. As in Section 11.4 we fix $s_0 \in \mathbf{C}$ such that $\mathrm{Re}(s_0) = 0$. We fix $V_0 \in \mathcal{V}_{\eta,s_0}$. Then there exists a section $V_s \in \mathcal{V}_{\eta,s}$ such that $V_{s_0} = V_0$. By (21.3) and (21.7) we have

$$w\hat{V}_{s_0}(b) = \lim_{s\to s_0} \int k_{s,\eta}(a,b)\hat{V}_s(a)da, \tag{21.9}$$

where the limit is taken on a compact path to s_0 coming from the right. Now we use Theorem 21.2 and bounds on $k_{s,\eta}(a,b)$ in Section 11 exactly as in Section 11.4 to apply the dominated convergence theorem in (21.9) and get

$$w\hat{V}_{s_0}(b) = \int k_{s_0,\eta}(a,b)\hat{V}_{s_0}(a)da.$$

21.3. DISCRETE SERIES. For the discrete series we use Lemma 21.3, and the bounds on $k_{s,\eta,\delta}(a,b)$ from Section 11 to apply the dominated convergence theorem in (21.6) and get (21.7).

21.4. GENERAL FOURIER TRANSFORM. We can generalize (21.4) in the following way. Let $\lambda \in \mathbf{R}^*$ and let $\hat{f}^\lambda$ be the λ-Fourier transform of f as defined in (2.2). Let $d_\lambda x$ be the measure defined in (2.1). Then

$$\hat{f}(y) = |\lambda|^{-1/2}\hat{f}^\lambda(y/\lambda). \tag{21.10}$$

Therefore if $V = V_f$ is as above, we have

$$|\lambda|^{-1/2}w_0\hat{V}^\lambda(b/\lambda) = |\lambda|^{-1}\int k_{s,\eta}(a,b)\hat{V}^\lambda(a/\lambda)d_\lambda a.$$

Hence

$$w_0\hat{V}^\lambda(b) = |\lambda|^{1/2}\int k_{s,\eta}(\lambda a,\lambda b)\hat{V}^\lambda(a)d_\lambda a.$$

We define

$$k_{\lambda,\eta,s}(a,b) = |\lambda|^{1/2}k_{s,\eta}(\lambda a,\lambda b). \tag{21.11}$$

Then for $V = V_f$ as above we have

$$w_0\hat{V}^\lambda(b) = \int k_{\lambda,\eta,s}(a,b)V^\lambda(a)da. \tag{21.12}$$

21.5. A KERNEL FORMULA IN THE WHITTAKER MODEL. We now translate this kernel formula into the Whittaker model. For every $f \in \pi_{\eta,s}$ we define

$$W_{f,\lambda}(g) = \int f(w_0n(x)g)e^{-2\pi i\lambda x}d_\lambda x.$$

The integral converges or is defined by analytic continuation. We have

$$
\begin{aligned}
W_f(t(a)) &= \int f(w_0 n(x) t(a)) e^{-2\pi i\lambda x} d_\lambda x \\
&= \int f(t(a^{-1}) z(a) w_0 n(xa^{-1})) e^{-2\pi i\lambda x} d_\lambda x \\
&= (\operatorname{sgn}(a))^\eta |a|^{1/2-s} \int f(w_0 n(x)) e^{-2\pi i\lambda a x} d_\lambda x \\
&= (\operatorname{sgn}(a))^\eta |a|^{1/2-s} \hat{V}_f^\lambda(a).
\end{aligned} \tag{21.13}
$$

Now rewriting (21.12) using the above formula we have

$$(\operatorname{sgn}(b)^\eta)|b|^{s-1/2} W_{f,\lambda}(t(b)w_0) = \int k_{\lambda,\eta,s}(a,b)(\operatorname{sgn}(a))^\eta |a|^{s-1/2} W_{f,\lambda}(t(a)) d_\lambda a.$$

Writing this in the form

$$W_{f,\lambda}(t(b)w_0) = \int J_{\lambda,\eta,s}(a,b) W_{f,\lambda}(t(a)) d^* a$$

we get that

$$J_{\lambda,\eta,s}(a,b) = (\operatorname{sgn}(ab))^\eta |a/b|^{s-1/2} k_{\lambda,\eta,s}(a,b).$$

Thus

$$
\begin{aligned}
J_{\eta,s}(a,b) =& (-1)^\eta |\lambda| \operatorname{sgn}(ab))^\eta |ab|^{1/2} \\
&\times [e^{\frac{(-1-2s)(-\operatorname{sgn}(\lambda a)-\operatorname{sgn}(\lambda b))\pi i}{4}} K_{2s}(4\pi|\lambda|\sqrt{|ab|} e^{\frac{(-\operatorname{sgn}(\lambda a)+\operatorname{sgn}(\lambda b))\pi i}{4}}) \\
&+ e^{\frac{(-1-2s)(\operatorname{sgn}(\lambda a)+\operatorname{sgn}(\lambda b))\pi i}{4}} K_{2s}(4\pi|\lambda|\sqrt{|ab|} e^{\frac{(\operatorname{sgn}(\lambda a)-\operatorname{sgn}(\lambda b))\pi i}{4}})].
\end{aligned} \tag{21.4}
$$

It is easy to check that:

LEMMA 21.4: *$J_{\lambda,s,\eta}(a,b) = Q(ab)$ for some function Q, that is, $J_{\lambda,s,\eta}(a,b) = J_{\lambda,s,\eta}(ab,1)$.*

We define a function $j_{\lambda,\eta,s}$: $Bw_0B \to \mathbf{C}$ by

$$
\begin{aligned}
j_{\lambda,\eta,s}(t(x)w_0) &= J_{\lambda,\eta,s}(x,1), \\
j_{\lambda,\eta,s}(n(y)z(c)t(x)w_0 n(z)) &= e^{2\pi i(x+y)} j_{\eta,s}(t(x)w_0).
\end{aligned}
$$

We have

$$
j_{\lambda,\eta,s}(t(x)w_0) = \tag{21.15}
$$

$$
\begin{cases}
2(-1)^\eta |\lambda|^{1/2}|x|^{1/2}(e^{\frac{(1+2s)\pi i}{2}} - e^{\frac{(-1-2s)\pi i}{2}}) K_{2s}(4\pi|\lambda|\sqrt{|x|}) \\
\text{if } x > 0; \\
2|\lambda|^{1/2}|x|^{1/2}(K_{2s}(4\pi|\lambda| i\sqrt{|x|}) + K_{2s}(-4\pi|\lambda| i\sqrt{|x|})) \\
= |x|^{1/2} \frac{\pi}{\sin(2\pi s)}(-e^{s\pi i} - e^{-s\pi i})(J_{2s}(4\pi|\lambda|\sqrt{|x|}) - J_{-2s}(4\pi|\lambda|\sqrt{|x|})) \\
\text{if } x < 0.
\end{cases}
$$

That is

$$(21.16)\qquad j_{\lambda,\eta,s}(t(x)w_0) = \begin{cases} 4(-1)^\eta|\lambda|^{1/2}|x|^{1/2}\cos(s\pi)K_{2s}(4\pi|\lambda|\sqrt{|x|}) & \text{if } x>0; \\ |\lambda|^{1/2}|x|^{1/2}\frac{-\pi}{\sin(\pi s)}(J_{2s}(4\pi|\lambda|\sqrt{|x|}) - J_{-2s}(4\pi|\lambda|\sqrt{|x|})) & \text{if } x<0. \end{cases}$$

22. Appendix 3: Fourier transform and relative Bessel functions

We recall some formulas from [6]:

$$\int_0^\infty x^{-1/2}K_{2\nu}(ax^{1/2})\cos(xy)dx =$$
$$-\frac{\pi^{3/2}}{4\sqrt{|y|}\cos(\nu\pi)}\Big(J_\nu\Big(\frac{a^2}{8|y|}\Big)\sin\Big(\frac{\nu\pi}{2}-\frac{a^2}{8|y|}-\frac{\pi}{4}\Big) + Y_\nu\Big(\frac{a^2}{8|y|}\Big)\cos\Big(\frac{\nu\pi}{2}-\frac{a^2}{8|y|}-\frac{\pi}{4}\Big)\Big),$$

$$\int_0^\infty x^{-1/2}K_{2\nu}(ax^{1/2})\sin(xy)dx =$$
$$-\operatorname{sgn}(y)\frac{\pi^{3/2}}{4\sqrt{|y|}\cos(\nu\pi)}\Big(J_\nu\Big(\frac{a^2}{8|y|}\Big)\cos\Big(\frac{\nu\pi}{2}-\frac{a^2}{8|y|}-\frac{\pi}{4}\Big) - Y_\nu\Big(\frac{a^2}{8|y|}\Big)\sin\Big(\frac{\nu\pi}{2}-\frac{a^2}{8|y|}-\frac{\pi}{4}\Big)\Big),$$

$$\int_{-\infty}^0 |x|^{-1/2}J_\nu(a|x|^{1/2})\cos(xy)dx = \int_0^\infty x^{-1/2}J_\nu(ax^{1/2})\cos(xy)dx$$
$$= \frac{\pi^{1/2}}{\sqrt{|y|}}\cos\Big(-\frac{\nu\pi}{4}+\frac{a^2}{8|y|}-\frac{\pi}{4}\Big)J_{\nu/2}\Big(\frac{a^2}{8|y|}\Big),$$

$$\int_{-\infty}^0 |x|^{-1/2}J_\nu(a|x|^{1/2})\sin(xy)dx$$
$$= -\int_0^\infty x^{-1/2}J_\nu(ax^{1/2})\sin(xy)dx$$
$$= \operatorname{sgn}(y)\frac{\pi^{1/2}}{\sqrt{|y|}}\sin\Big(-\frac{\nu\pi}{4}+\frac{a^2}{8|y|}-\frac{\pi}{4}\Big)J_{\nu/2}\Big(\frac{a^2}{8|y|}\Big).$$

Consequently we have

$$(22.1)\qquad \int_{-\infty}^0 |x|^{-1/2}J_\nu(a|x|^{1/2})e^{ixy}dx = \frac{\pi^{1/2}}{\sqrt{|y|}}e^{\operatorname{sgn}(y)(\frac{a^2}{8|y|}-\frac{\nu\pi}{4}-\frac{\pi}{4})i}J_{\nu/2}\Big(\frac{a^2}{8|y|}\Big),$$

$$\int_0^\infty x^{-1/2}K_{2\nu}(ax^{1/2})e^{ixy}dx = -\frac{\pi^{3/2}}{4\sqrt{|y|}\cos(\nu\pi)} \times \Big(\operatorname{sgn}(y)ie^{\operatorname{sgn}(y)(\frac{a^2}{8|y|}-\frac{\nu\pi}{2}+\frac{\pi}{4})i}J_\nu\Big(\frac{a^2}{8|y|}\Big) + e^{\operatorname{sgn}(y)(\frac{a^2}{8|y|}-\frac{\nu\pi}{2}+\frac{\pi}{4})i}Y_\nu\Big(\frac{a^2}{8|y|}\Big)\Big).$$

Now define

$$\tilde{f}(y) = \int_0^\infty f(x)e^{ixy}dx,$$

where dx is the standard Lebesgue measure, and

$$\hat{f}^\lambda(y) = \int_0^\infty f(x)e^{2\pi i\lambda xy}d_\lambda x.$$

Then we have

$$\hat{f}^\lambda(y) = |\lambda|^{1/2}\tilde{f}(2\pi\lambda y).$$

Hence we have

$$\begin{aligned}
&(-1)^\eta\int_0^\infty 4|\lambda|^{1/2}\cos(s\pi)x^{-1/2}K_{2s}(4\pi|\lambda|x^{1/2})e^{2\pi i\lambda xy}d_\lambda x\\
=&4(-1)^{\eta+1}|\lambda|\cos(s\pi)\frac{\pi^{3/2}}{4\sqrt{2\pi|\lambda||y|}\cos(s\pi)}\\
&\times\Big(\operatorname{sgn}(\lambda y)ie^{\operatorname{sgn}(\lambda y)(\frac{\pi|\lambda|}{|y|}-\frac{s\pi}{2}+\frac{\pi}{4})i}J_s\Big(\frac{\pi|\lambda|}{|y|}\Big)\\
&\quad+e^{\operatorname{sgn}(\lambda y)(\frac{\pi|\lambda|}{|y|}-\frac{s\pi}{2}+\frac{\pi}{4})i}Y_s\Big(\frac{\pi|\lambda|}{|y|}\Big)\Big)\\
=&(-1)^{\eta+1}\frac{|\lambda|^{1/2}\pi}{\sqrt{2|y|}}\Big(\operatorname{sgn}(\lambda y)ie^{\operatorname{sgn}(\lambda y)(\frac{\pi|\lambda|}{|y|}-\frac{s\pi}{2}+\frac{\pi}{4})i}J_s\Big(\frac{\pi|\lambda|}{|y|}\Big)\\
&\quad+e^{\operatorname{sgn}(\lambda y)(\frac{\pi|\lambda|}{|y|}-\frac{s\pi}{2}+\frac{\pi}{4})i}Y_s\Big(\frac{\pi|\lambda|}{|y|}\Big)\Big)\\
=&(-1)^{\eta+1}\frac{|\lambda|^{1/2}\pi}{\sqrt{2|y|}\sin(\pi s)}e^{\frac{\pi\lambda i}{y}}\\
&\times\Big[(\operatorname{sgn}(\lambda y)ie^{\operatorname{sgn}(\lambda y)(-\frac{s\pi i}{2}+\frac{\pi i}{4})}\sin(\pi s)+e^{\operatorname{sgn}(\lambda y)(-\frac{s\pi i}{2}+\frac{\pi i}{4})}\cos(\pi s))\\
&\times J_s\Big(\frac{\pi|\lambda|}{|y|}\Big)-e^{\operatorname{sgn}(\lambda y)(-\frac{s\pi i}{2}+\frac{\pi i}{4})}J_{-s}\Big(\frac{\pi|\lambda|}{|y|}\Big)\Big]\\
=&(-1)^{\eta+1}\frac{|\lambda|^{1/2}\pi}{\sqrt{2|y|}\sin(\pi s)}e^{\frac{\pi\lambda i}{y}}\Big(e^{\operatorname{sgn}(\lambda y)(\frac{s\pi i}{2}+\frac{\pi i}{4})}J_s\Big(\frac{\pi|\lambda|}{|y|}\Big)\\
&\quad-e^{\operatorname{sgn}(\lambda y)(-\frac{s\pi i}{2}+\frac{\pi i}{4})}J_{-s}\Big(\frac{\pi|\lambda|}{|y|}\Big)\Big)
\end{aligned}$$

and

$$
\begin{aligned}
&\int_{-\infty}^{0} |\lambda|^{1/2}\frac{-\pi}{\sin(\pi s)}|x|^{1/2}(J_{2s}(4\pi|\lambda|\sqrt{|x|}) - J_{-2s}(4\pi|\lambda|\sqrt{|x|}))e^{2\pi i\lambda xy}d_\lambda x\\
&= |\lambda|\frac{-\pi}{\sin(\pi s)}\frac{\pi^{1/2}}{\sqrt{2\pi|\lambda||y|}}\Big(e^{\operatorname{sgn}(\lambda y)(\frac{\pi|\lambda|}{|y|}-\frac{s\pi}{4}-\frac{\pi}{4})i}J_s\Big(\frac{\pi|\lambda|}{|y|}\Big)\\
&\qquad\qquad - e^{\operatorname{sgn}(y)(\frac{\pi|\lambda|}{|y|}+\frac{s\pi}{4}-\frac{\pi}{4})i}J_{-s}\Big(\frac{\pi|\lambda|}{|y|}\Big)\Big)\\
&= \frac{-|\lambda|^{1/2}\pi}{\sin(\pi s)\sqrt{2|y|}}e^{\frac{\pi\lambda i}{y}}\Big(e^{\operatorname{sgn}(\lambda y)(-\frac{s\pi}{2}-\frac{\pi}{4})i}J_s\Big(\frac{\pi|\lambda|}{|y|}\Big) - e^{\operatorname{sgn}(\lambda y)(\frac{s\pi}{2}-\frac{\pi}{4})i}J_{-s}\Big(\frac{\pi|\lambda|}{|y|}\Big)\Big).
\end{aligned}
$$

Hence using the formulas in (21.16) and the above formulas we get that

$$
\begin{aligned}
&\int_{-\infty}^{\infty}|x|^{-1}j_{\lambda,\eta,s}(t(x)w_0)e^{2\pi i\lambda xy}d_\lambda x\\
&=\frac{|\lambda|^{1/2}\pi}{\sin(\pi s)\sqrt{2|y|}}e^{\frac{\pi\lambda i}{y}}\Big[((-1)^{\eta+1}e^{\operatorname{sgn}(\lambda y)(\frac{s\pi i}{2}+\frac{\pi i}{4})} - e^{\operatorname{sgn}(\lambda y)(-\frac{s\pi i}{2}-\frac{\pi i}{4})})J_s\Big(\frac{\pi|\lambda|}{|y|}\Big)\\
&\qquad + ((-1)^{\eta}e^{\operatorname{sgn}(\lambda y)(\frac{-s\pi i}{2}+\frac{\pi i}{4})} + e^{\operatorname{sgn}(\lambda y)(\frac{s\pi i}{2}-\frac{\pi i}{4})})J_{-s}\Big(\frac{\pi|\lambda|}{|y|}\Big)\Big]\\
&=\frac{|\lambda|^{1/2}\pi}{\sin(\pi s)\sqrt{2|y|}}e^{\frac{\pi\lambda i}{y}}e^{\operatorname{sgn}(\lambda y)(\frac{\pi i}{4}}\Big[((-1)^{\eta+1}e^{\operatorname{sgn}(\lambda y)\frac{s\pi i}{2}}\\
&\qquad + \operatorname{sgn}(\lambda y)ie^{\operatorname{sgn}(\lambda y)(-\frac{s\pi i}{2})})J_s\Big(\frac{\pi|\lambda|}{|y|}\Big)\\
&\qquad + ((-1)^{\eta}e^{\operatorname{sgn}(\lambda y)(\frac{-s\pi i}{2})} - \operatorname{sgn}(\lambda y)ie^{\operatorname{sgn}(\lambda y)(\frac{s\pi i}{2})})J_{-s}\Big(\frac{\pi|\lambda|}{|y|}\Big)\Big].
\end{aligned}
$$

22.1. Discrete Series. If $\eta = 1$ and $s = d - 1/2$ with $d \in \mathbf{N}$, then we have

$$
(22.2)\qquad j_{\lambda,\eta,s}(t(x)w_0) = \begin{cases} |\lambda|^{1/2}|x|^{1/2}\frac{-\pi}{\sin(\pi s)}(J_{2d-1}(4\pi|\lambda|\sqrt{|x|}) - J_{-(2d-1)}(4\pi|\lambda|\sqrt{|x|})) & \text{if } x < 0;\\ 0 & \text{if } x > 0.\end{cases}
$$

Since $\sin(\pi s) = (-1)^{d-1}$ and $J_{-(2d-1)}(z) = -J_{2d-1}(z)$, we get that

$$
(22.3)\qquad j_{\lambda,\eta,s}(t(x)w_0) = \begin{cases} (-1)^d 2|\lambda|^{1/2}\pi|x|^{1/2}J_{2d-1}(4\pi|\lambda|\sqrt{|x|}) & \text{if } x < 0;\\ 0 & \text{if } x > 0.\end{cases}
$$

Hence we have

$$\begin{aligned}
&\int_{-\infty}^{0}(-1)^d 2|\lambda|^{1/2}\pi|x|^{-1/2}J_{2d-1}(4\pi\lambda\sqrt{|x|})e^{2\pi i\lambda xy}d_\lambda x\\
&=(-1)^d\frac{|\lambda|^{1/2}\pi\sqrt{2}}{|y|^{1/2}}e^{\frac{\lambda\pi i}{y}}e^{\operatorname{sgn}(\lambda y)\frac{-d\pi i}{2}}J_{d-1/2}(\pi|\lambda|/|y|)\\
&=(\operatorname{sgn}(y)i)^d\frac{\sqrt{2}|\lambda|^{1/2}\pi}{|y|^{1/2}}e^{\frac{\lambda\pi i}{y}}J_{d-1/2}(\pi|\lambda|/|y|).
\end{aligned}$$

23. Appendix 4: General theory of Bessel-like distributions

Bessel-like distributions on a reductive group G were defined by Gelfand and Kazhdan [8] and by Shalika [17]. We shall define the same distributions using a "trace class" approach.

Let H be a separable Hilbert space with an inner form $<,>$. Let λ,β be linear functionals on H (not necessarily bounded). We say that (λ,β) is a summable pair if there exists an orthonormal basis f_i, $i=1,2\ldots$, such that

$$\sum_{i,j}|\lambda(f_i)\beta(f_j)|<\infty. \tag{23.1}$$

LEMMA 23.1: *Assume that (λ,β) is a summable pair. Let $e_i, i=1,2\ldots$, be an orthonormal basis. Then*

$$S(\lambda,\beta)=\sum_i\lambda(e_i)\overline{\beta(e_i)} \tag{23.2}$$

converges absolutely and is independent of the orthonormal basis $\{e_i\}$.

Proof: Writing $e_i=\sum<e_i,f_j>f_j$ we formally have

$$\sum_i\lambda(e_i)\overline{\beta(e_i)}=\sum_{i,j,k}\lambda(f_j)\overline{\beta(f_k)}<e_i,f_j><f_k,e_i>. \tag{23.3}$$

We would like to show that the right hand side is absolutely convergent. We have

$$\sum_i|<e_i,f_j><f_k,e_i>|\le\left(\sum_i|e_i,f_j>|^2\right)^{1/2}\left(\sum_i|f_k,e_i>|^2\right)^{1/2}=1,$$

hence

$$\sum_{i,j,k}|\lambda(f_j)\overline{\beta(f_k)}<e_i,f_j><f_k,e_i>|\le\sum_{j,k}|\lambda(f_j)\overline{\beta(f_k)}|<\infty,$$

since (λ, β) is a summable pair. This implies that (23.3) holds and that the left hand side of (23.3) converges absolutely. To show that it is independent of the basis we use the following equality:

$$\begin{aligned} \sum_k \sum_i \sum_j \lambda(f_j)\overline{\beta(f_k)} < e_i, f_j >< f_k, e_i > &= \sum_k \sum_i \lambda(e_i) < f_k, e_i > \overline{\beta(f_k)} \\ &= \sum_k \lambda(f_k)\overline{\beta(f_k)}. \end{aligned} \tag{23.4}$$

∎

COROLLARY 23.2: *If (λ, λ) is a summable pair, then λ is a bounded linear functional on H.*

Proof: By Lemma 23.1, $S(\lambda, \lambda) = \sum_i |\lambda(e_i)|^2 < \infty$ and this sum is independent of the orthonormal basis $\{e_i\}$. Assume that λ is not bounded. Then there exists a sequence of unit vectors $v_n \in H$, $n = 1, 2 \dots$ such that $|\lambda(v_n)| > n$. Since each vector v_n can be part of an orthonormal basis, it follows that $S(\lambda, \lambda) > n^2$ for every positive integer n. Hence $S(\lambda, \lambda) = \infty$, a contradiction. ∎

23.1. Bessel-like distributions. We shall now apply the previous discussion to the case of admissible representations of reductive groups over local fields.

Let k be a local field and let G be the k points of a reductive group defined over k. Let dg be a fixed Haar measure on G. The space $C_c^\infty(G)$ is the space of compactly supported, smooth functions on G if k is archimedean, or the space of compactly supported, locally constant functions if k is non-archimedean. Let (π, H) be an irreducible admissible representation of G on a Hilbert space H. For $\phi \in C_c^\infty(G)$ we define the bounded linear operator $\pi(\phi)$: $H \to H$ by

$$\pi(\phi)v = \int_G \phi(g)\pi(g)v dg.$$

Let $H_\infty = \{\pi(\phi)v | f \in C_c^\infty(G), v \in H\}$. If k is archimedean then we topologize H_∞ in the usual way. If k is non-archimedean then we give H_∞ the discrete topology.

Let λ be a continuous functional on H_∞ and $\phi \in C_c^\infty(G)$. We define the functional λ_ϕ on H (see [17]) by

$$\lambda_\phi(v) = \lambda(\pi(\phi)v), \quad v \in H.$$

LEMMA 23.3: *Let λ, β be continuous linear functionals on H_∞. Let $\phi_1, \phi_2 \in C_c^\infty(G)$. Then $(\lambda_{\phi_1}, \beta_{\phi_2})$ is a summable pair.*

Proof: If k is non-archimedean, then $\pi(\phi_1)$ and $\pi(\phi_2)$ are of finite rank and the proof is immediate.

Assume k is archimedean. We shall assume for the moment that G is linear and connected. For the general case the proof will follow the arguments in [23] 8.1. Let K be a maximal compact in G and $\mathfrak{k}$ be the Lie algebra of K. Let $\mathcal{Z}(\mathfrak{k}_{\mathbf{C}})$ be the center of the universal enveloping algebra of $\mathfrak{k}_{\mathbf{C}}$, the complexification of $\mathfrak{k}$. We write $H_K = \bigoplus_\tau H_\tau$, where H_K is the space of K finite vectors in H, τ is an irreducible, finite-dimensional representation of K and H_τ is the τ isotypic component of H_K. Since π is admissible, H_τ is a finite-dimensional space. Let $\{v_i\}$ be an orthonormal basis for H_τ. For $z \in \mathcal{Z}(\mathfrak{k}_{\mathbf{C}})$, we denote by $\chi_\tau(z)$ the value of the character of τ on z. We have

$$\begin{aligned}
\sum_{i,j} |\lambda_{\phi_1}(v_i)\beta_{\phi_2}(v_j)| &= \sum_{i,j} |\lambda(\pi(\phi_1)v_i)\beta(\pi(\phi_2)v_j)| \\
&= |\chi_\tau(z)|^{-1} | \sum_{i,j} |\lambda(\pi(\phi_1)\pi(z)v_i)\beta(\pi(\phi_2)v_j)| \\
&= |\chi_\tau(z)|^{-1} | \sum_{i,j} |\lambda(\pi(z\phi_1)v_i)\beta(\pi(\phi_2)v_j)| \\
&\leq |\chi_\tau(z)|^{-1} \dim(H_\tau)^2 ||\lambda_{z\phi_1}|| ||\beta_{\phi_2}||.
\end{aligned}$$

The proof now follows from the proof of [13] Theorem 10.2 or [23] Lemma 8.1.1. ∎

COROLLARY 23.4 ([17] Proposition 3.2): *Let λ be a continuous functional on H_∞ and $\phi \in C_c^\infty(G)$. Then the functional λ_ϕ is a bounded linear functional on H.*

Proof: By Lemma 23.3, $(\lambda_\phi, \lambda_\phi)$ is a summable pair, hence by Corollary 23.2, λ_ϕ is bounded. ∎

We define the distribution $B_{\lambda,\beta}$ by

$$B_{\lambda,\beta}(\phi_1, \phi_2) = S(\lambda_{\phi_1}, \beta_{\phi_2}), \phi_1, \phi_2 \in C_c^\infty(G).$$

Let $\phi_1, \phi_2 \in C_c^\infty(G)$ and $x \in G$. Define

$$R_x(\phi_1)(g) = \phi(gx), \quad \tilde{\phi}_1(g) = \phi_1(g^{-1})$$

and

$$\phi_1 * \phi_2(x) = \int_G \phi_1(g)\phi_2(g^{-1}x)dg.$$

Assume now that π is unitary (and irreducible). Let $<,>$ be a G invariant inner product on H. Since π is unitary, it follows that if $\{e_i\}$ is an orthonormal basis for H, then $\{\pi(x)e_i\}$ is also an orthonormal basis of H. Therefore by Lemma 23.1 we have $B_{\lambda,\beta}(R_x\phi_1, R_x\phi_2) = B_{\lambda,\beta}(\phi_1, \phi_2)$ for all $x \in G$. Hence there exists a unique distribution $J_{\lambda,\beta}$ on G such that

$$B_{\lambda,\beta}(\phi_1, \phi_2) = J_{\lambda,\beta}(\phi_1 * \tilde{\phi}_2). \tag{23.5}$$

We call $J_{\lambda,\beta}$ a Bessel-like distribution. A distribution of this type has been constructed in [17] p. 184. We now recall this construction and show that it agrees with our construction. The construction in [17] works for general admissible representations (as opposed to unitary representations in our case). It is possible to extend the discussion here to such representations. However, since we are only interested in unitary representations we leave this construction as an exercise. A hint can be found in [1] Lemma 6.1.

Recall that by the Riesz representation theorem, every bounded linear functional λ on H is associated to a unique $v_\lambda \in H$ such that

$$\lambda(u) =< u, v_\lambda >, \quad u \in H.$$

LEMMA 23.5: *Let (λ, β) be a summable pair of bounded linear functionals. Let $S(\lambda, \beta)$ be defined as in (23.2). Then*

$$S(\lambda, \beta) =< v_\beta, v_\lambda > .$$

Proof: Let $\{e_i\}$ be an orthonormal basis for H. Then $v_\lambda = \sum < v_\lambda, e_i > e_i = \sum \overline{\lambda(e_i)} e_i$ and $v_\beta = \sum < v_\beta, e_i > e_i = \sum \overline{\beta(e_i)} e_i$. Hence

$$< v_\beta, v_\lambda >= \sum \lambda(e_i)\overline{\beta(e_i)}. \qquad \blacksquare$$

COROLLARY 23.6: $B_{\lambda,\beta}(\phi_1, \phi_2) =< v_{\beta_{\phi_2}}, v_{\lambda_{\phi_1}} >$.

COROLLARY 23.7: $J_{\lambda,\beta}(\phi) = \lambda(v_{\beta_{\check{\phi}}}) = \overline{\beta(v_{\lambda_\phi})}$.

Proof: We have that

$$J_{\lambda,\beta}(\phi_1 * \tilde{\phi}_2) = B_{\lambda,\beta}(\phi_1, \phi_2) =< v_{\beta_{\phi_2}}, v_{\lambda_{\phi_1}} >= \lambda_{\phi_1}(v_{\beta_{\phi_2}}) = \lambda(\pi(\phi_1)v_{\beta_{\phi_2}}).$$

It is easy to check that $\pi(\phi_1)v_{\beta_{\phi_2}} = v_{\beta_{(\phi_1*\tilde{\phi}_2)^\frown}}$, hence we get the first equality. In a similar way we get that

$$J_{\lambda,\beta}(\phi_1 * \tilde{\phi}_2) = \overline{\beta(v_{\lambda_{(\phi_2*\tilde{\phi}_1)^\frown}})}.$$

Since $(\phi_2 * \tilde{\phi}_1)^\frown = \phi_1 * \tilde{\phi}_2$ we get the second equality. ∎

Using the above form for J we can express J as a sum as in Lemma 23.5. Let $\{e_i\}$ be an orthonormal basis of smooth vectors. Then

$$J_{\lambda,\beta}(f) = \sum \lambda(\pi(f)e_i)\overline{\beta}(e_i). \tag{23.6}$$

Remark 23.8: Assume that k is archimedean. It is easy to see, as in [17] p. 184, that the distribution $J_{\lambda,\beta}$ is an eigendistribution for the center of the universal enveloping algebra of $\mathfrak{g}$, the Lie algebra of G.

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